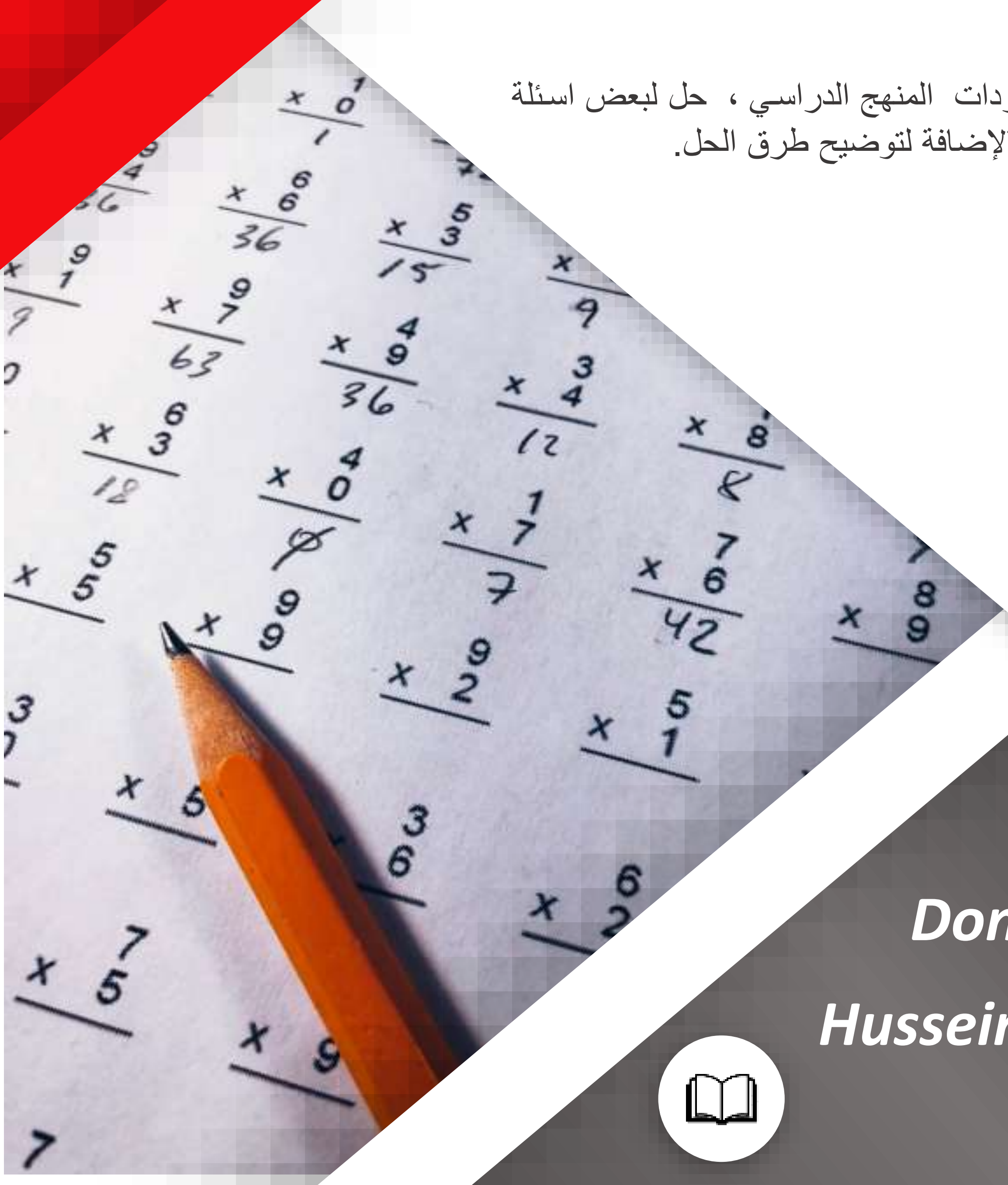
Engineering analysis

توضيح لمفردات المنهج الدراسي ، حل لبعض اسئلة السنوات ، بالإضافة لتوضيح طرق الحل.



Done by

Hussein Amjad

المقدمة

* تعد مادة التحليلات الهندسية من المواد الأساسية ذات الارتباط الواضح بالمواد الدراسية الأخرى .

* المصاعب التي التواجه الطالب بسبب تكديس المادة و عدم الإحساس بحجم المنهج (المادة مطولة لكن سهلة وبسيطة) .

* سنتناول كل موضوع   (شرح الموضوع ، معرفة الطريقة المناسبة للحل ، كيفية تفسير السؤال و حله ، أمثلة و أسئلة عامة)

sullabus

First-order differential equations *

Second-order differential equations *

Solutions of trential equation by series *

Laplace transforms *

First-order differential equations

First order differential equation

D.E. = Differential equation (هي المعادلات التي يكون فيها المتغير دالة ، حيث تظهر العلاقة بين الدالة ومشتقتها)

Ordinary D.E :-

There is only one independent variables

ex) $\frac{dy}{dx} + y = 5$

where : y : dependent variables

x : independent variables

Order : المرتبة

(أعلى أس للمشتقة)

(أكبر مشتقة)

Degree : الدرجة

(أعلى قوة لأعلى اس للمشتقة)

ex)

$$\left(\frac{d^2y}{dx^2}\right)^2 + 4x\frac{dy}{dx} + 2y = \cos x$$

(2nd order, 2nd degree)

liner D-E

It's equation of the "1st" degree in the dependent viraiable and it's derivates. (...تفهم من الأمثلة)

ex) $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 4y = \cos x$$

2nd order, 1st degree, Non Liner

لان الحد التابع تربيع

ex) $$\frac{d^3y}{dx^3} - \frac{dy^2}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 2x^3\frac{dy}{dx} + 4x = 4e^x$$

3rd order, 1st degree, Non Liner

ex) $$\frac{d^2y}{dx^2} + 4y\frac{dy}{dx} + 2y = \cos x$$

2nd order, 1st degree - Non Liner

لأن المقام مضروب

بـ y (independent)

ex) $$\frac{d^2y}{dx^2} + \sin y$$

2nd order, 1st degree, Non Liner

حتى تكون Liner

يجب أن تكون sin x

Solution of 1^{st} order D-E

هنالك العديد من الطرق لحل المعادلات التفاضلية من الدرجة الأولى.

ولكن حسب دراستنا سنتطرق إلى خمسة طرق

1) Separable 1^{st} order D.E.

- if the D.E is of the form:-

$$f(y)\,dy = g(x)\,dx \quad \text{--- (1)}$$

- The solution will be

$$\int f(y)\,dy = \int g(x)\,dx + C \quad \text{--- (2)}$$

ex) Solve: $y\,dx - x\,dy = x(dy - y\,dx)$

الحل بطريقة الفصل نعزل dx عن dy و نعزل المتغيرات كذلك ومن بعدها نكامل.

$$x\,dy + x\,dy = xy\,dx + y\,dx$$

$$2x\,dy = y(x+1)\,dx$$

- Divide both side by (x, y) $\Rightarrow \dfrac{2\,dy}{y} = \dfrac{(x+1)\,dx}{x}$
- Now integrate both sides $\Rightarrow 2\int \dfrac{dy}{y} = \int \dfrac{x+1}{x}\,dx$

$$\Rightarrow 2\int \frac{dy}{y} = \int dx + \int \frac{1}{x}\,dx$$

$$\Rightarrow 2\ln y = x + \ln x + C$$

2) exact 1^{st} order D.E

If The D.E is of the form :-

$$\left[M(x,y)\,dx + N(x,y)\,dy = 0 \right] - (1)$$

So The D.E is exact if and only is

$$\frac{dm}{dy} = \frac{dN}{dx} \quad --- (2)$$ حفظ

Eq (2) may be written :-

$$u = \frac{du}{dx}dx + \frac{du}{dy}dy = 0 \quad ---(3)$$

Compare eq (3) and (5)

$$\frac{du}{dx} = M \Rightarrow du = M\,dx$$

$$\Rightarrow \int du = \int M\,dx + g(y)$$

$$\Rightarrow u = \int M\,dx + g(y) \quad \ldots \text{——} (4)$$

Differentiate eq.(4) with respect to (y)

$$\frac{du}{dy} = \frac{d}{dy}\int M\,dx + \frac{dg}{dy} \quad --- (5), \quad \because \frac{du}{dy} = N$$

$$\therefore N = \frac{d}{dy} M\,dx + \frac{dg(y)}{dy}$$

$$\frac{dg(y)}{dy} = N - \frac{d}{dy}\int M\,dx$$

$$\Rightarrow dg(y) = \left[N - \frac{d}{dy}\int M\,dx\right] dy$$

$$\Rightarrow \int dg(y) = g(y) = \int \left[N - \frac{d}{dy}\int M\,dx\right] dy \quad \text{---(6)}$$

Subtite eq(6) into eq(4)

$$\Rightarrow \quad u = \int M\,dx + \int N\,dy - \int \frac{d}{dy}\left(\int M\,dx\right) dy$$

* الأشتقاق للفهم وليس للحفظ ، لتركز بالأشتقاق شوف المثال بعده

وراح تتوضح الصورة (الطريقة عبارة عن تسفيط)

ex) Solve :- $(2xy + 3y^3)\,dx + (x^2 + 9xy^2)\,dy = 0$

Sol :- $M\,dx + N\,dy = 0$

$M = 2xy + 3y^3 \quad , \quad N = x^2 + 9xy^2$

$\frac{dM}{dy} = 2x + 9y^2 \quad , \quad \frac{dN}{dx} = 2x + 9y^2$

$\because \frac{dM}{dy} = \frac{dN}{dx} \quad , \therefore$ The D.E is exact

let; $u = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = 0$

$\frac{du}{dx} = M = 2xy + 3y^3 \Rightarrow du = (2xy + 3y^3)\,dx$

$\Rightarrow \quad u = x^2y + 3xy^3 + g(y) \quad \text{---(1}$

Differential eq (1) with respect to (y)

$$\frac{du}{dy} = x^2 + 9xy^2 + \frac{dg(y)}{dy} \quad — ②$$

But $$\frac{du}{dy} = N = x^2 + 9xy^2 \quad -- ③$$

$$\Rightarrow \cancel{x^2} + \cancel{9xy^2} + \frac{dg(y)}{dy} = \cancel{x^2} + \cancel{9xy^2}$$

$$\Rightarrow \frac{dg(y)}{d(y)} = 0$$

$$d(g(y)) = 0 \Rightarrow \int dg(y) = 0$$

$$\because g(y) = 0 \Rightarrow \therefore u = x^2 y + 3xy^3 + C = 0$$

* من حل المثال الواضح شرط حل ال exact هو $\frac{dM}{dy} = \frac{dN}{dx}$
في حال عدم تحقيق هذا الشرط سنلجئ لتغيرات في المعادلة
من سبيل تحقيقه يعرف هذا التغيير او القيمة الدخيلة على المعادلة

(Intigration factor)

if $\frac{dM}{dy} \neq \frac{dN}{dx}$ the D.E is non exact:

The integration factor (R_x) or $(R_{(y)})$ must be

$$\boxed{R_x} = e^{\int \left(\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} \right) dx} \qquad \boxed{R_y} = e^{\int \left(\frac{\frac{dM}{dy} - \frac{dN}{dx}}{-M} \right) dy}$$

أي قيمة ناتجة من أحد القوانين
نضربها بالمعادلة راح تصحح وتحقق الشرط

تستعمل أحدى القانونين بشرط ان يكون ناتج Rx او Ry سهل
(للتصير ذكي وتاخذ قانون يطلع الناتج ماله معقد ترى تتورط).

ex) Solve $(x^2+x-y)\,dx + x\,dy = 0$ --- ①

Sol) $M\,dx + N\,dy = 0$

$M = x^2+x-y$, $N = x$

$\frac{dM}{dy} = -1$, $\frac{dN}{dx} = 1$

$\frac{dM}{dy} \neq \frac{dN}{dx}$, So D.E is Non exact

يعني نحتاج معامل تكميلي

$Rx = e^{\int \frac{-1-1}{x} dx}$

$\Rightarrow Rx = e^{\int -\frac{2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}}$

$\Rightarrow Rx = x^{-2} = \frac{1}{x^2}$

multiply eq ① with Rx

$\left(1+\frac{1}{x}-\frac{y}{x^2}\right)dx + \frac{1}{x}\,dy = 0$ (هسة كل شغلنا راح يكون على هذه المعادلة)

$M = 1+\frac{1}{x}-\frac{y}{x^2}$, $N = \frac{1}{x}$

$\frac{dM}{dy} = \frac{-1}{x^2}$, $\frac{dN}{dx} = \frac{-1}{x^2}$

So, $\frac{dM}{dy} = \frac{dN}{dx}$ and The D.E is exact

بعدها يستمر الحل وفق قاعدة الـ exact .

3) Liner first order D.E

$$\frac{dy}{dx} + P(x)\, y = Q(x) \quad \text{---}(1)$$

The integration factor

$$R_x = e^{\int P(x)\, dx} \quad \text{----}(2)$$

$$y = \frac{1}{R(x)} \int R_x\, Q(x) + \frac{C}{R_x} \quad \text{--}(3$$

* المعادلات للحفظ

Ex) Solve : $x^2 dy + (2xy - x + 1)\, dx = 0$

Sol Divided by dx

$$x^2 \frac{dy}{dx} + 2xy - x + 1 = 0$$

Divided by x^2

$$\Rightarrow \frac{dy}{dx} + 2\frac{y}{x} - \frac{1}{x} + \frac{1}{x^2} = 0$$

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x} - \frac{1}{x^2}$$

$$\because \frac{dy}{dx} + P(x)\, y = Q(x)$$

$$\therefore P(x) = \frac{2}{x}, \quad Q(x) = \frac{1}{x} - \frac{1}{x^2}$$

$$R(x) = e^{\int P(x)\, dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$y = \frac{1}{R(x)} \int R_x Q_x + \frac{C}{R(x)}$$

$$= \frac{1}{x^2} \int (x-1) + \frac{C}{x^2} = \frac{1}{x^2}\left(\frac{x^2}{2} - x\right) + \frac{C}{x^2} = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$$

4) Homogenous first order D.E.

$$\frac{dy}{dx} = f(x, y) = f(\lambda x, \lambda y) \quad \text{--- (1)}$$

where λ = constant (معامل تجانسي

or $$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{--- (2)}$$ – The D-E is homogenous.

Let: $$y = vx \quad \text{--- (3)}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{--- (4)}$$

substitute in Eq (2)

$$\Rightarrow v + x\frac{dv}{dx} = f(v)$$

$$x\frac{dv}{dx} = f(v) - v \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v} \quad \text{--- (5)}$$

(بالمختصر نحول كل y إلى vx وكل $\frac{dy}{dx}$ إلى $v+x\frac{dv}{dx}$

ونحل بعد ترتيب المعادلة بطريقة الفصل والتكامل وبعد ما نخلص

من الحل نحول كل v إلى $\frac{y}{x}$)

ex) Solved: $(3y^3 - x^3)dx = 3xy^2\, dy$

Sol: divide by (dx)

$$3y^3 - x^3 = 3xy^2 \frac{dy}{dx} \qquad \div 2xy^2$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{3}\frac{x^2}{y^2} = \frac{y}{x} - \frac{1}{3}\frac{1}{\left(\frac{y}{x}\right)^2} \quad \text{--- (1)}$$

So D.E is homogenous --, Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Sub sition is eq (1)

$$V + x\frac{dv}{dx} = \frac{y}{x} - \frac{1}{3}\frac{x^2}{y^2}$$

$$\Rightarrow \cancel{V} + x\frac{dv}{dx} = \cancel{V} - \frac{1}{3}\frac{1}{V^2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1}{3V^2} \Rightarrow -3V^2 dV = \frac{dx}{x}$$

$$\Rightarrow -3\int v^2 dv = \int \frac{dx}{x}$$

$$\Rightarrow -\cancel{3}\frac{V^3}{\cancel{3}} = \ln|x| + C \Rightarrow -V^3 = \ln x + C$$

$$\Rightarrow -\frac{y^3}{x^3} = \ln x + c \Rightarrow -y^3 = x^3\ln x + x^3 c$$

$$\Rightarrow y = \sqrt[3]{x^3(\ln x + c)}$$

5) Bernouli equation :-

If is of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \qquad n \neq 1 \quad \text{--- (1)}$$

Divide by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

Let $Z = \dfrac{1}{y^{n-1}}$

Then, the D.E is changed to liner 1^{st} order D.E.

[يابة برنولي هو عبارة عن لينير ، الله وفقها وزادت y أكثر]

ex) $$xy - \frac{dy}{dx} = y^4 e^{\frac{-3x^2}{2}} \quad \text{--- (1)}$$

Sol :- $$\frac{dy}{dx} - xy = -e^{\frac{-3x^2}{2}}\, y^4$$

E.q (1) is Bernoull eq....

$$P(x) = -x \quad , \quad Q(x) = -e^{\frac{-3x^2}{2}} \qquad \div y^4$$

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{x}{y^3} = -e^{\frac{-3x^2}{2}} \quad \text{--- (2)}$$

Let $Z = \dfrac{1}{y^3} \Rightarrow dZ = -3\dfrac{1}{y^4}\, dy$ --- (3)

$$\frac{dz}{dy} = \frac{-3}{y^4} \Rightarrow \frac{dy}{dZ} = -\frac{y^4}{3} \quad \text{--- (4)}$$

But, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{y^4}{3} * \frac{dz}{dx}$ — (5)

Sub in eq. (2)

$\left(\frac{-1}{3} \frac{dz}{dx} - \frac{x}{y^3} = -e^{\frac{3x^2}{2}} \right) * -3$

$\frac{dz}{dx} + 3x\, z = 3 e^{\frac{-3x^2}{2}}$ — (6)

$P(x) = 3x \quad , \quad Q(x) = 3 e^{\frac{-3x^2}{2}}$

$R_x = e^{\int 3x\, dx} = e^{\frac{3x^2}{2}}$

$z = \frac{1}{R_x} \int R_x\, Q(x)\, dx + \frac{c}{R_x}$

$z = \frac{1}{e^{\frac{3x^2}{2}}} \int e^{\frac{3x^2}{2}}\, 3 e^{\frac{-3x^2}{2}} + \frac{c}{e^{\frac{3x^2}{2}}}$

$z = 3x\, e^{\frac{-3x^2}{2}} + c\, e^{\frac{-3x^2}{2}}$

$z = (3x + c)\, e^{-\frac{3x^2}{2}}$

$\frac{1}{y^3} = (3x - c)\, e^{\frac{-3x^2}{2}}$

$y^3 = \frac{e^{\frac{3x^2}{2}}}{(3x + c)}$

{ ملاحظات حول كيفية معرفة الطريقة التي ستستخدم في الحل

* يكون اختيار طريقة حل المعادلة عن طريق خطوات سنتطرق اليها

* هذه الخطوات عن اجتهاد شخصي وغير مرتبطة بمصدر معين.

- اختيار First order : -

أولاً - Homogenous / عند اختيار المعادلة المتجانسة نعوضها مكان كل y بـ x فاذا كانت الاسس جميعها متساوية في المعادلة فالمعادلة تكون متجانسة ، ويجب الانتباه إذا كان في المعادلة دالة مثلثية او دالة اس وان الزاوية أو الأساس $\left(\frac{y}{x}\right) = (V)$ فاذا كان غير ذلك فأن المعادلة بنسبة كبيرة غير متجانسة

ثانياً :- separable : - عند وجود (x) مع (y) يمكن فصلهم بطريقة ما (قسمة ، عامل مشترك) فالمعادلة لا يمكن حلها إلا بهذه الطريقة

ثالثاً :- Liner ، عند توفر الصيغة $\frac{dy}{dx} + P(x)y = Q(x)$

* وجود (y) مضروب في (x) ولا يمكن فصله

رابعاً :- Bernouli / نفس صيغة Liner لكن يوجد y^n مضروب $Q(x)$

خامساً :- exact ، ما بقرا بس هذا حله بالاخير بهذي الطريقة وخليها يم الله .....

حلول بعض أسئلة السنوات !-

supplementary * 15/9/2007 * Q 5

Q5. Solve the following differential equation by any method you suggested.

متجانسة

$$(x\sin\frac{y}{x} - y\cos\frac{y}{x})dx + x\cos\frac{y}{x}dy = 0$$

Let $y = v$

(10 Marks)

$$(x\sin\frac{y}{x} - y\cos\frac{y}{x})dx + x\cos(\frac{y}{x})dx = 0$$

$\div\ x\cos(\frac{y}{x})dx$

$$\Rightarrow \tan(\frac{y}{x}) - \frac{y}{x} + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan(\frac{y}{x})$$

Let $y = vx \Rightarrow v = \frac{y}{x}$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} - \tan(v)$$

$$\int\frac{dx}{x} = -\int\frac{\cos(v)}{\sin(v)}dv$$

$$\ln x = -\ln\sin(v) + C$$

ناخذ e للطرفين

$e^c = c_1$

$$x = \frac{1}{\sin(\frac{y}{x})} + c_1$$

2nd Term * 31/5/2008 * Q 5

Q5. Solve the following differential equation by any method you suggested.

$$(3x^2 - 6xy)dx - (3x^2 + 2y)dy = 0$$

(10 Marks)

$(3x^2 - 6xy)\,dx - (3x^2 + 2y)\,dy = 0$

Sol/ $M = 3x^2 - 6xy$, $N = -3x^2 - 2y$

$\frac{dm}{dy} = -6x$, $\frac{dN}{dx} = -6x$

$\frac{dm}{dy} = \frac{dN}{dx}$ $\therefore$ The D.E is exact

$\therefore u = \frac{du}{dx}dx + \frac{du}{dy}dy$

$\therefore \frac{du}{dx} = M = 3x^2 - 6xy$

$\Rightarrow du = (3x^2 - 6xy)\,dx \Rightarrow \int du = \int (3x^2 - 6xy)\,dx$

$\Rightarrow u = x^3 - 3x^2y + g(y)$

$\frac{du}{dy} = -3x^2 + \frac{dg(y)}{dy}$, $\because \frac{du}{dy} = N$

$\Rightarrow -3x^2 - 2y = -3x^2 + \frac{dg(y)}{dy}$

$dg(y) = -2y\,dy$

$\int dg(y) = \int -2y\,dy \Rightarrow g(y) = -y^2$

$\therefore u = x^3 - 3x^2y - y^2$

1^{st} term × 21/1/2006 × Q4

Q4. Show that the following 1^{st} order differential equation is homogeneous and find the solution.

$(xe^{\frac{y}{x}} + y)dx - xdy = 0$ (25 Mark)

$(xe^{y/x} + y)\,dx - x\,dx = 0 \qquad \div x\,dx$

$e^{y/x} + \frac{y}{x} - \frac{dy}{dx} = 0$

$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}$

Let $v = \frac{y}{x}$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \cancel{v} + x\frac{dv}{dx} = e^{v} + \cancel{v}$

$\Rightarrow \int \frac{dx}{x} = \int e^{-v}\,dv$

$\Rightarrow \ln x = -e^{\frac{-y}{x}} + c$

First Term * 2/3/2008 * Q2

Q4. Show that the following 1[st] order differential equation is homogeneous and find the solution.

$$(x+y)\,dy+(x-y)\,dx=0 \longrightarrow$$

H.W

(30 Mark)

$$(x+y)\,dy+(x-y)\,dx=0 \qquad \div x$$

$$\left(1+\frac{y}{x}\right)dy+\left(1-\frac{y}{x}\right)dx=0 \qquad \div\left(1+y/x\right)dx$$

$$\frac{dy}{dx}+\frac{1-\frac{y}{x}}{1+\frac{y}{x}}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y/x-1}{1-y/x}$$

Let $y=vx \Rightarrow v=y/x$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow x\cdot\frac{dv}{dx}+v=\frac{v-1}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{1+v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-1-v^2}{1+v} \Rightarrow \int\frac{-dx}{x}=\int\frac{1-v}{1+v^2}\,dv$$

$$\Rightarrow -\int\frac{dx}{x}=\int\frac{1}{1+v^2}\,dv+\frac{1}{2}\int\frac{2v}{1+v^2}\,dv$$

$$-\ln x=\tan^{-1}(v)+\frac{1}{2}\ln(1+v^2)+c$$

$$-\ln x=\tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right)+c$$

حلول Home work (1)

1- $x\,dy = (y^2 - 3y + 2)\,dx$

$\Rightarrow \frac{1}{x}dx = \frac{1}{(y^2-3y+2)}dy$

تكامل بالتجزئة

$\frac{1}{(y-2)(y-1)} = \frac{A}{y-2} + \frac{B}{y-1}$

$\Rightarrow \frac{1}{x}dx = \frac{1}{(y-2)(y+1)}dy$

$1 = A(y-1) + B(y-2)$

Let, $y = 1$

$\Rightarrow \int \frac{1}{x}dx = \int\left(\frac{1}{y-2} + \frac{-1}{y-1}\right)dy \Rightarrow B = -1$

Let; $y = 2$

$\Rightarrow \ln x + \ln c = \ln y-2 - \ln(y-1) \Rightarrow A = 1$

$\Rightarrow \ln xc = \ln\left(\frac{y-2}{y-1}\right) \Rightarrow xc = \frac{y-2}{y-1}$

$\Rightarrow xcy - xc = y - 2 \Rightarrow 2 - xc = y - xcy$

$\Rightarrow y(1 - xc) = 2 - xc \Rightarrow y = \frac{2-xc}{1-xc}$

2) $ye^{x+y}\,dx = dx$

$\Rightarrow ye^x e^y\,dy = dx \Rightarrow \int ye^y\,dy = \int e^{-x}\,dx$

$\Rightarrow ye^y - e^y = -e^{-x} + c$

تكامل udv

$\int u\,dv = uv - \int v\,du$

$\Rightarrow e^y(y-1) + e^{-x} = c$

$\int ye^y dy = ye^y - e^y$

3) $(xy^2 - y)\,dx + x(xy - 1)\,dy = 0$

$(xy^2 - y)\,dx + (x^2y - x)\,dy = 0$ —— (1

$M\,dx + N\,dy = 0$

$M = (xy^2 - y)$, $N = (x^2y - x$

$\frac{dM}{dy} = 2x - 1$, $\frac{dN}{dx} = 2x - 1$

$\therefore \frac{dM}{dy} = \frac{dN}{dx}$ ∴ The D.E is exact

$u = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = 0$ —— (2

on comparision (1 and (2

$\Rightarrow \frac{du}{dx} = x^2y - y \Rightarrow du = xy^2 - y\,dx$

$\int du = u = \frac{x^2y^2}{2} - xy + g(y)$ —— (3

Differentiale eq (3 with respect to (y)

$\Rightarrow \frac{du}{dy} = x^2y - x + g(y)$ —— (4

from eq (3 $N = x^3y - x = \frac{du}{dy}$ —— (5

Equating Eq (4 and (5

$\Rightarrow \cancel{x^2y} - x = \cancel{x^2y} - x + \frac{dg(y)}{dy}$

$\Rightarrow \frac{dg(y)}{dy} = 0 \Rightarrow dg = 0 \Rightarrow \int dg = 0 \Rightarrow g = C$ sub. in Eq (3

$\Rightarrow \frac{x^2y^2}{2} - xy + C = 0$

5) $(x^2-2y^2)\,dx - xy\,dy = 0$

$\Rightarrow (x^2+2y^2)\,dx = xy\,dy$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+2y^2}{xy} \Rightarrow \frac{dy}{dx} = \frac{x}{y} + \frac{2y}{x}$ — ①

Let $y = xv \Rightarrow v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ — ②

From ① and ② $\Rightarrow \frac{1}{v} + 2v = v + x\frac{dv}{dx}$

$\Rightarrow x\frac{dv}{dx} = \frac{1}{v} + v \Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v}$

$\Rightarrow \frac{v}{1+v^2}\,dv = \frac{dx}{x} \Rightarrow \frac{1}{2}\ln(1+v^2) = \ln x + \ln c$

$\Rightarrow \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln xc$

$\Rightarrow \sqrt{1+\frac{y^2}{x^2}} = xc$

$\Rightarrow 1+\frac{y^2}{x^2} = x^2c_1 \Rightarrow y = x\sqrt{x^2c_1 - 1}$

6) $(xe^{\frac{y}{x}} + y)dx - x\,dy = 0$

$\Rightarrow (xe^{\frac{y}{x}} + y)dx = x\,dy$

$\frac{dy}{dx} = \frac{xe^{y/x}}{x} + \frac{y}{x} \Rightarrow \frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x}$ ①

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ — ②

from ① and ②

$\Rightarrow e^{v} + \cancel{v} = \cancel{v} + x\frac{dv}{dx}$

$\Rightarrow \frac{dv}{dx} = \frac{e^{v}}{x}$

$\Rightarrow \int \frac{1}{x}dx = \frac{1}{e^{v}}dv$

$\Rightarrow \ln x = -e^{-v} \Rightarrow \ln x = -e^{\frac{y}{x}} + c$

$\Rightarrow \ln x + e^{-\frac{y}{x}} = C$

7) $(x+y)\,dy + (x-y)\,dx = 0$

تم حله ضمن أسئلة بعض السنوات

8) $\left(x \sin\frac{y}{x} - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0$

تم حله ضمن أسئلة بعض السنوات

9) $\cosh(x)\,dy + (y\sinh(x) + e^{x})\,dx = 0$

$\Rightarrow (y\sinh(x) + e^{x})\,dx = -\cosh(x)\,dy$

$$\frac{-y\sinh(x)}{\cosh(x)} - \frac{e^{x}}{\cosh(x)} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} + \frac{\sinh(x)}{\cosh(x)}y = \frac{-e^{x}}{\cosh(x)}$$ D.E is Linear

$$P = \frac{\sinh(x)}{\cosh(x)}, \quad Q = \frac{e^{x}}{\cosh(x)}$$

$$R_x = e^{\int P\,dx}$$

$$R_x = e^{\int \frac{\sinh(x)}{\cosh(x)}dx} = e^{\ln\cosh(x)} = \cosh(x)$$

$$R_x\, y = \int R_x Q\,dx + C$$

$$y\cosh(x) = \int \frac{-e^{x}}{\cancel{\cosh(x)}} * \cancel{\cosh(x)}\cdot dx + C$$

$$y = \frac{-e^{x} + C}{\cosh(x)}$$

10) $(1+x^2)(dy-dx) = 2xy\,dx$

$$\frac{dy-dx}{dx} = \frac{2xy}{1+x^2} \Rightarrow \frac{dy}{dx} - 1 = \frac{2xy}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{2x}{1+x^2}\right) y = 1$$

$$\frac{dy}{dx} - P(x)\,y = Q(x).$$

$$\therefore \quad P(x) = \frac{-2x}{1+x^2}$$

$$Rx = e^{\int P(x)\,dx} = e^{\int \frac{-2x}{1+x^2}dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$$

$$Q(x) = 1$$

$$y = \frac{1}{Rx}\int Rx \cdot Q(x) + \frac{c}{Rx}$$

$$\Rightarrow y = 1+x^2 \int \frac{1}{1+x^2}(1)\,dx + c(1+x^2)$$

$$\Rightarrow y = (1+x^2)\tan^{-1}(x) + c(1+x^2)$$

11) $x \cdot \frac{dy}{dx} + (1+x)y = e^{-x}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = e^{-x} * \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} + P(x) = Q(x)$

$R_x = e^{\int P(x)\,dx} = e^{\int\left(\frac{1+x}{x}\right)dx} = e^{\ln x + x}$

$\Rightarrow R_x = e^{\ln x} * e^{x} = x e^{x}$

$y = \frac{1}{R_x}\int R(x)\,Q(x)\,dx + \frac{C}{R_x}$

$y = \frac{1}{x e^{x}}\int \frac{\cancel{x}\,\cancel{e^{x}}}{1} * \cancel{e^{-x}} * \frac{1}{\cancel{x}}\,dx + \frac{C}{x e^{x}}$

$y = \frac{1}{\cancel{x} e^{x}} * \cancel{x} + \frac{C}{x e^{x}}$

$y = e^{-x}\left(1 + \frac{C}{x}\right)$

12) $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$

$\frac{dy}{dx} + P(x) = Qx$

$Px = \frac{1}{1-x}$

$R_x = e^{\int P(x)\,dx} = e^{\int \frac{1}{1-x}} = e^{-\ln(1-x)} = \frac{1}{1-x}$

$y = \frac{1}{R_x} \int R_x * Q_{(x)}\,dx + \frac{c}{R_x}$

$y = \frac{1}{(1/(1-x))} \int \frac{1}{1-x} * x^2 - x\,dx + \frac{c}{(\frac{1}{1-x})}$

$y = (1-x)(\frac{-x^2}{2}) + (1-x)c$

$\Rightarrow y = (1-x)(c - \frac{x^2}{2})$

13) $\frac{dy}{dx} - \frac{2}{x} y = y^3$ — (1)

$\frac{dy}{dx} + P(x) y = Q(x) y^n$, $n = 3$

Multiply by y^{-3}

$y^{-3} \frac{dy}{dx} - \frac{2}{x} y^{-2} = 1$

Let $Z = y^{-2} \Rightarrow \frac{dz}{dx} = -2 y^{-3} \frac{dy}{dx}$ — (2)

Sub (2) in (1) and multibly by (-2)

$\frac{dz}{dx} + \frac{4}{x} z = -2$

$R_x = e^{\int P(x) dx} = e^{4 \ln x} \Rightarrow x^4$

$Z = \frac{1}{R_x} \int R(x) Q(x) dx + \frac{C}{R_x}$

$Z = \frac{1}{x^4} \int x^4 * -2 \, dx + \frac{C}{x^4}$

$Z = \frac{1}{\cancel{x^4}} * \frac{-2}{5} x^{\cancel{5}\,1} + \frac{C}{x^4}$

$Z = y^{-2} \Rightarrow Z = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{Z}}$

$y = \frac{\sqrt{5} \, x^2}{\sqrt{5C - 2x^4}}$

(4) $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$

$M = 4xy + 3y^2 - x$, $N = x^2 + 2xy$

$\frac{dm}{dy} = 4x + 6y$, $\frac{dN}{dx} = 2x + 2y$

$\frac{dm}{dy} \neq \frac{dN}{dy}$ Non exact

$$Rx = e^{\int \frac{dm/dy - dN/dx}{N} dx}$$

$$Rx = e^{\int \frac{4x + 6y - 2x - 2y}{x^2 + 2xy} dx}$$

$$Rx = e^{\int \frac{2x + 4y}{x^2 + 2xy} dx} = e^{\int \frac{2\cancel{(x+2y)}}{x\cancel{(x+2y)}} dx}$$

$$\Rightarrow Rx = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

multiply by I. factor (x^2)

$$(4x^3y + 3y^2x^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$$

$M = 4x^3y + 3y^2x^2 - x^3$, $N = x^4 + 2x^3y$

$\frac{dm}{dy} = 4x^3 + 6x^2y$, $\frac{dN}{dx} = 4x^3 + 6x^2y$

$\frac{dM}{dy} = \frac{dN}{dx}$ equ. is exact

15) $y(2x+y)\,dx + (3x^2+4xy-y)\,dy = 0$

$M = 2xy + y^2 \qquad , \; N = 3x^2 + 4xy - y$

$\frac{dM}{dy} = 2x - 2y \qquad , \; \frac{dN}{dx} = 6x + 4y$

$\frac{dM}{dy} \neq \frac{dN}{dx}$

$$R_y = e^{\int \frac{\frac{dN}{dx} - \frac{dM}{dy}}{M} dy}$$

$$R_y = e^{\int \frac{6x+4y-2x-2y}{2xy+y^2} dy}$$

$$\Rightarrow R_y = e^{\int \frac{4x+2y}{2xy+y^2} dy} \Rightarrow R_y = e^{\int \frac{2(\cancel{2x+y})}{y(\cancel{2x+y})} dy}$$

$$\Rightarrow R_y = e^{2\ln y} = y^2$$

multibly by (y^2)

$(2xy^3+y^4)\,dx + (3x^2y^2 + 4xy^3 - y^3)\,dy = 0$

$\frac{dM}{dy} = 6xy^2 + 4y^3 \qquad , \; \frac{dN}{dx} = 6xy^2 + 4y^3$

$\frac{dM}{dy} = \frac{dN}{dx}$

$\frac{du}{dx} = M \Rightarrow du = 2xy^3 + y^4 \quad dx$

$u = \frac{2}{2}x^2y^3 + xy^4 + g(y)$

$$\frac{du}{dy} = 3y^2x^2 + 4xy^3 + \frac{g(y)}{dy}$$

$$\because N = \frac{du}{dy}$$

$$\therefore \quad 3\cancel{x^2y^2} + \cancel{4xy^3} - y^3 = \cancel{3y^2x^2} + \cancel{4xy^3} + \frac{g(y)}{dy}$$

$$g(y) = -y^3\,dy = -1/4\, y^4$$

$$\Rightarrow u = x^2y^3 + xy^4 - \frac{1}{4}y^4$$

16) $\frac{dy}{dx} + xy = xy^2$

$$\frac{dy}{dx} + P(x)\,y = Qx\,y^2$$

divided by y^2

$$\Rightarrow y^{-2}\frac{dy}{dx} + xy^{-1} = x$$

Let $Z = y^{-1} \Rightarrow \frac{dy}{dx} = -y^{-2}$, Multiply by (-1) and sub (Z)

$$\Rightarrow \frac{dZ}{dx} - xZ = -x \quad , \quad P(x) = -x \, , \, Q = -x$$

$$R(x) = e^{\int P(x)} = e^{\int -x} = e^{-\frac{x^2}{2}}$$

$$Z = \frac{1}{Rx}\int Rx\, Q(x) + \frac{C}{Rx} = \frac{1}{e^{-x^2/2}}\int e^{-\frac{x^2}{2}}(-x)\,dx + \frac{C}{e^{-\frac{x^2}{2}}}$$

$$Z = \frac{1}{e^{-x^2/2}}\left(e^{\frac{-x^2}{2}}\right) + C\,e^{x^2/2} \, , \; Z = 1 - Ce^{\frac{x^2}{2}}$$

$$y^{-1} = 1 + Ce^{\frac{x^2}{2}} \Rightarrow y = \frac{1}{1 + Ce^{x^2/2}}$$

17) $x \cdot \frac{dy}{dx} + y = \frac{1}{y^2}$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x} y^{-2}$ (multply by y^2)

$y^2 \frac{dy}{dx} + \frac{1}{x} y^3 = \frac{1}{x}$

Let : $Z = y^3 \Rightarrow \frac{dZ}{dx} = 3y^2$

multibly by (3) and sub (Z)

$\frac{dZ}{dx} + \frac{3}{x} z = \frac{3}{x}$, $P(x) = \frac{3}{x}$, $Q(x) = \frac{3}{x}$

$R(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$

$Z = \frac{1}{R(x)} \int R(x)\, Q(x)\, dx + \frac{C}{R(x)}$

$Z = \frac{1}{x^3} \int x^3 * \frac{3}{x} dx + \frac{C}{x^3}$

$Z = \frac{1}{x^3} (x^3) + \frac{C}{x^3} \Rightarrow y^3 = 1 + \frac{C}{x^3}$

18) $dy + 2xy\,dx = x e^{-x^2} y^3\,dx$

$\Rightarrow \dfrac{dy}{dx} + 2xy = x e^{-x^2} y^3$

multiply by y^{-3}

$\Rightarrow y^{-3}\dfrac{dy}{dx} + 2xy^{-2} = x e^{-x^2}$

$Z = y^{-2} \Rightarrow \dfrac{dz}{dx} = -2y^{-3}$

Multiply by (-2) and sub. (Z)

$\dfrac{dz}{dx} - 4xz = -2x e^{-x^2}$, $P(x) = -4x$, $Q(x) = -2x e^{-x^2}$

$R(x) = e^{\int P(x)\,dx} = e^{\int -4x\,dx} = e^{-2x^2}$

$z = \dfrac{1}{Rx} \int Rx\, Q(x) + \dfrac{c}{Rx}$

$\Rightarrow Z = \dfrac{1}{e^{-2x^2}} \int e^{-2x^2} * -2x e^{-x^2} + \dfrac{c}{e^{-2x^2}}$

$\Rightarrow Z = e^{2x^2} \int -2x e^{-3x^2}\,dx + c e^{2x^2}$

$\Rightarrow Z = \dfrac{1}{3} e^{-x^2} + C e^{2x^2}$

$y^2 = \dfrac{1}{\frac{1}{3} e^{-x^2} + C e^{2x^2}}$

بعض الأسئلة الخارجية لاختبار فهمك للمادة .

1) $\frac{dy}{dx}\sqrt{2xy}=1$

ans: $\frac{3}{\sqrt{2}}y^{\frac{3}{2}}-2\sqrt{x}=c$

2) $2x\,dx-dy=x(x\,dy-2y\,dx),\quad y(-3)=\frac{1}{2}$

ans: $\ln(1+x^2)+\ln(1-y)=1.6093$

3) $\frac{dy}{dx}=\sqrt{\frac{(x+y+xy+1)x}{x}}$

ans: $2\sqrt{y+1}-\frac{2}{3}(1+x)^{\frac{3}{2}}=c$

4) $(x+y\cot\frac{x}{y})\,dy-y\,dx=0$

ans: $\ln x+\ln(\cos\frac{x}{y})+\ln\frac{y}{x}=c$

5) $(3y^3-x^3)\,dx=3xy^2\,dy,\quad y(1)=2$

ans: $\ln x+\frac{1}{3}\frac{y^3}{x^3}=\frac{8}{3}$

6) $\frac{dy}{dx}=\frac{x+y}{x-y}$

$\ln(x+2)-\frac{1}{2}\ln(\frac{y-2}{x+2}+1)+\frac{1}{2}\ln(\frac{y-2}{x+2}-1)$
$+\frac{1}{2}\ln(\frac{(y-2)^2}{(x+1)^2}-1)=c$

7) $\cosh(x)\,dy+(y\sinh(x)+e^x)\,dx=0$

ans: $\cosh(x)*y=-e^x+c$

Second-order differential equations

Second_ order Differential equations

Second-order D.E.

- non liner
  - خالية من x
  - خالية من y
  - متجانسة Homogenous
- Liner
  - $y''f(x) + y'(fx_1) + f(x_2)y = g(x)$

    $g(x) = 0$

    وبهذه الحالة يكون الحل $y = y_c$

    بالصفحات التالية موجود كيفية ايجاد y
  - $y''f(x_1) + y'(fx_1) + yf(x_3) = g(x)$

    $g(x) \neq 0$

    الحل العام $y = y_c + y_p$

    سنتطرق إلى ايجاد الحل العام في الصفحات التالية .

المخطط فيه رسم مبسط لأنواع المعادلات التفاضلية من الدرجة الثانية ، سنتطرق إلى شرح كل نوع على حدة

• وصدك نسيت حلو بيدكم مو بس تياوعون ترى هاي تحليلات مو جريدة .

* تقسيم Liner عندما تكون المعاملات ثابتة

اما إذا كانت المعاملات غير ثابتة تُحل وفق المنهج بطريقة (Euler وفق المنهج يعني بلا كيو غير طرق .

* Non Line (خالية من y)

Let $\frac{dy}{dx} = P$ —— ①

$\frac{d^2y}{dx^2} = \frac{dP}{dx}$ —— ②

The eq is changed to 1^{st} order

(حول حسب معادلة ① و ② وبعدها حل كأنها درجة أولى

من تخلص رجع قيمة P إلى $\frac{dy}{dx}$ وبعدها كاملها

بطريقة الفصل separable وكان الله غفور رحيم)

ex; $\frac{d^2y}{dx^2} + x\frac{dy}{dx} = ax$ —— ①

ans; Let: $\frac{dy}{dx} = P$, $\frac{d^2y}{dx^2} = \frac{dP}{dx}$

sub into eq ①

$\frac{dP}{dx} + xP = ax$ (liner, first order)

$P(x) = x$, $Q(x) = ax$

$R(x) = e^{\int x\,dx} = e^{\frac{x^2}{2}}$

$P = e^{-\frac{x^2}{2}} \int e^{\frac{x^2}{2}} (ax)\,dx + c_1 e^{-\frac{x^2}{2}}$

$P = a e^{-\frac{x^2}{2}} (e^{\frac{x^2}{2}}) + c_1 e^{-\frac{x^2}{2}}$

$$P = a e^{-\frac{x^2}{2}} \left(c e^{\frac{x^2}{2}} \right) + c_1 e^{-\frac{x^2}{2}}$$

$$P = a + c_1 e^{-\frac{x^2}{2}}$$

$$\frac{dy}{dx} = a + c_1 e^{-\frac{x^2}{2}} \Rightarrow \int dy = \int \left(a + c_1 e^{-\frac{x^2}{2}}\right) dx$$

$$\Rightarrow \quad y = ax + c_1 \int e^{-\frac{x^2}{2}} dx$$

Note :- error function $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$

$$\int_0^x e^{-x^2} dx = \sqrt{\frac{\pi}{2}} \operatorname{erf}(x)$$

$$\int e^{-\left(\frac{x}{\sqrt{2}}\right)^2} = \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y = ax + c_1 \frac{\sqrt{\pi}}{2} \operatorname{erf} \frac{x}{\sqrt{2}} + c_2$$

$$\Rightarrow \quad y = ax + C_3 \operatorname{erf} \frac{x}{\sqrt{2}} + C_2 \quad , \quad C_3 = C_1 \frac{\sqrt{\pi}}{2}$$

(مسألة erf(x) لتعطيها أهتمام لان راح تاخذها بالكورس

الثاني مجرد افهم طريقة حل السؤال)

* Non Liner (حل الحالة من x)

Let $P = \frac{dy}{dx} \Rightarrow \frac{dP}{dx} = \frac{d^2y}{dx^2} = \frac{dP}{dy} * \frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = P \frac{dP}{dy}$

The D.E is changed to 1st order (نفس طريقة حل الحالة من y)

ex: $y \frac{d^2y}{dx^2} + 1 = \left(\frac{dy}{dx}\right)^2$

Sol: $y P \frac{dP}{dy} + 1 = P^2$

$\Rightarrow y P \frac{dP}{dy} = P^2 - 1$

$\Rightarrow \frac{P\, dP}{P^2-1} = \frac{dy}{y}$

$\Rightarrow \frac{1}{2} \ln |P^2-1| = \ln |y| + \ln c$

$\Rightarrow \ln (P^2-1) = 2 \ln |y| + 2 \ln C$

$\ln (P^2-1) = \ln y^2 C^2$

$P^2 - 1 = C_1 y^2 \quad (C^2 = C_1)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 - 1 = C_1 y^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = C_1 y^2 + 1$

$\Rightarrow \frac{dy}{dx} = \sqrt{C_1 y^2 + 1} \Rightarrow \int \frac{dy}{\sqrt{C_1 y^2 + 1}} = \int dx$

$\sqrt{C_1 y^2+1} \longrightarrow \sqrt{C_1 \left(y^2 + \frac{1}{C_1}\right)}$

$\Rightarrow \frac{1}{\sqrt{C_1}} \int \frac{dy}{\sqrt{y^2 + \left(\frac{1}{\sqrt{C_1}}\right)^2}} = \int dx$

$\Rightarrow \frac{1}{\sqrt{C_1}} \sinh^{-1}(\sqrt{C_1}\, y) = x + C_2 \Rightarrow \sinh^{-1}\sqrt{C_1}\, y = C_3 x + C_4$

+ Homogenous 2nd order D.E

If the D.E is written in the form

$$x \frac{d^2y}{dx^2} = f\left(\frac{y}{x}, \frac{dy}{dx}\right) \quad --- (1$$

So, the 2nd order D.E is homogenous

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad (2)$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2} \quad (3)$$

Substitute in eq (1

$$x\left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) = f\left(v, v + x\frac{dv}{dx}\right)$$

$$\Rightarrow x^2\frac{d^2v}{dx^2} = f\left(v, v + x\frac{dv}{dx}\right) - 2x\frac{dv}{dx}$$

$$\Rightarrow x^2\frac{d^2v}{dx^2} = f_1\left(v, x\frac{dv}{dx}\right) \quad --- (4)$$

Let $x = e^t \Rightarrow t = \ln x \Rightarrow \frac{dt}{dx} = \frac{1}{x} \quad - (5)$

$$\frac{dv}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \frac{dv}{dt}\frac{1}{x} \quad -- (6)$$

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dv}{dt}\right)$$

$$\frac{d^2v}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dv}{dt}\right) - \frac{1}{x^2}\frac{dv}{dt}$$

$$= \frac{1}{x} \frac{dv}{dt} \left(\frac{dv}{dt}\right) \cdot \frac{dt}{dx} - \frac{1}{x^2} \frac{dv}{dt}$$

$$= \frac{1}{x^2} \frac{d^2v}{dt^2} - \frac{1}{x^2} \frac{dv}{dt} \quad \text{--- (7)}$$

sub eq (4)

$$x^2 \left(\frac{1}{x^2} \frac{d^2v}{dt^2} - \frac{1}{x^2} \frac{dv}{dt}\right) = f_1 \left(v, x \left\{\frac{1}{x} \frac{dv}{dt}\right\}\right)$$

$$\frac{d^2v}{dt^2} - \frac{dv}{dt} = f_1 \left(v, \frac{dv}{dt}\right) \quad \text{--- (8)}$$

eq (8) is D.E in which the independent variable doesn't

حاول تفهم الاشتقاق ولتلع وياه

(طريقة الحل راح تكون بتحويل كل

$$\frac{dy}{dx} \xRightarrow{\text{إلى}} v + x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} \xRightarrow{\text{إلى}} 2 \frac{dv}{dx} + x \frac{d^2v}{dx^2}$$

بعدها نرتب المعادلة ونحول كل

$$\frac{dv}{dx} \xRightarrow{\text{إلى}} \frac{1}{x} \frac{dv}{dt}$$

$$\frac{d^2v}{dx^2} \xRightarrow{\text{إلى}} \frac{1}{x^2} \frac{d^2v}{dt^2} - \frac{1}{x^2} \frac{dv}{dt}$$

وبعدها نحل ونرتب الحل وتطلعلنا معادلة من الدرجة الثانية بدلالة $\frac{dv}{dt}$ نحلها ، وبعدها بالأخير نرجع كل V إلى $\frac{y}{x}$.

Ex : $2x^2 y \frac{d^2y}{dx^2} + y^2 = x^2 \left(\frac{dy}{dx}\right)^2$

Sol : Dividing by $2xy$

$$\Rightarrow x \frac{d^2y}{dx^2} = -\frac{1}{2}\frac{y}{x} + \frac{1}{2}\frac{x}{y}\left(\frac{dy}{dx}\right)^2 \quad (1)$$

Then The D-E is homogeneous.

$$\therefore \frac{dy}{dx} = V + x\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{dv^2}{dx^2}$$

$$\therefore \cancel{x} \left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) = \frac{1}{2}\frac{1}{v}\left(V + x\frac{dv}{dx}\right)^2 - \frac{1}{2}v$$

$$\Rightarrow x\frac{d^2v}{dx^2} = \frac{1}{2}\left(\frac{1}{v}\right)\left(v^2 + 2vx\frac{dv}{dx} + x^2\left\{\frac{dv}{dx}\right\}^2\right)$$

$$- \frac{1}{2}v - 2x\frac{dv}{dx}$$

$$= \frac{1}{2}v + x\frac{dv}{dx} + \frac{x^2}{2v}\left(\frac{dv}{dx}\right)^2 - \frac{1}{2}v - 2x\frac{dv}{dx}$$

$$x^2\frac{d^2v}{dx} = \frac{x^2}{2v}\left(\frac{dv}{dx}\right)^2 - x\frac{dv}{dx} \quad (2)$$

$$\therefore \frac{dv}{dx} = \frac{1}{x}\frac{dv}{dt}, \quad \frac{d^2v}{dx^2} = \frac{1}{x^2}\frac{d^2v}{dt} - \frac{1}{x^2}\frac{dv}{dt}$$

$$\therefore \frac{d^2v}{dt^2} - \frac{dv}{dt} = \frac{x^2}{2v}\left(\frac{1}{x^2}\left(\frac{dv}{dt}\right)^2 - x\left(\frac{1}{x}\frac{dv}{dt}\right)\right)$$

$$\Rightarrow \frac{d^2v}{dt^2} = \frac{1}{2v}\left(\frac{dv}{dt}\right)^2 \quad (3)$$

$$\frac{dv}{dt} = P, \quad \frac{d^2v}{dt^2} = \frac{dP}{dt} = \frac{dP}{dv}\cdot\frac{dv}{dt} = P\frac{dP}{dv}$$

$$P\frac{dP}{dV} = \frac{1}{2V}P^2$$

$$\Rightarrow \frac{dP}{P} = \frac{dV}{V} \Rightarrow \int \frac{dP}{P} = \int \frac{dV}{V}$$

$$\ln P = \frac{1}{2}(\ln V + \ln C)$$

$$P = CV^{1/2} \Rightarrow \frac{dV}{dt} = C_1 V^{1/2}$$

$$\Rightarrow \frac{dV}{V^{1/2}} = C\,dt \Rightarrow \int \frac{dV}{V^{1/2}} = \int C_1\,dt$$

$$\Rightarrow \int V^{-1/2}\,dV = C_1 \int dt$$

$$\Rightarrow 2V^{1/2} = C_1 t + C_2$$

$$\Rightarrow 2\left(\frac{y}{x}\right)^{1/2} = C_1 t + C_2$$

$$\Rightarrow \left(\frac{y}{x}\right)^{1/2} = C_3 t + C_4 \qquad C_3 = \frac{C_1}{2}, \quad C_4 = \frac{C_2}{2}$$

$$\Rightarrow y = x(C_3 t + C_4)^2$$

* 2^{nd} order liner D-E . when $g(x)=0$

$$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \qquad -(1)$$

الحل العام $y = y_c$

The solution = complementery function

يتم إيجاد y_c بطريقتين

$$y_c = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$, un equal roots

* $A_1, A_2 \rightarrow$ ثوابت و (m_1, m_2) $\rightarrow$ نجدها بتحليل المعادلة *

ex$_1$: Solve : $\frac{d^2y}{dx^2}$ $5\frac{dy}{dx}+6y=0$

Sol; $m^2-5m+6=0$ (نحول كل $\frac{dy}{dx}$ إلى m)

$\Rightarrow (m-3)(m-2)=0 \Rightarrow m_1=3,\ m_2=2$

$\therefore\ y_c = A_1 e^{3x} + A_2 e^{2x}$.

$\because\ y=y_c \quad \therefore\ y = A_1 e^{3x} + A_2 e^{2x}$

ex$_2$: Solve : $y''-4y'+5y=0$

Sol : $m^2-4m+5=0$ (بالدستور)

$$m_1, m_2 = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2\pm i$$

$$\therefore \; y = A_1 e^{(2+i)x} + B e^{(2-i)x}$$ (ركز شلون نحول i)

$$\Rightarrow y = e^{2x}\left(A_1 e^{ix} + A_2 e^{-ix}\right)$$

$$\Rightarrow y = e^{2x}\left(A_1 \cos x + A_1 i \sin x + A_2 \cos x - A_2 i \sin x\right)$$

$$\Rightarrow y = e^{2x}\left((A_1 + A_2)\cos x + (iA_1 - iA_2)\sin x\right)$$

$$\Rightarrow y = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$$

b) equal Roots ($m_1 = m_2$)

$$y = y_c = (A + Bx)\, e^{mx}$$

ex; solve: $y'' + 6y' + 9y = 0$

ans: $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$, $m_1 = m_2 = -3$

$$y = y_c = (A + Bx)\, e^{-3x}$$

* 2^{nd} order liner D-E , when $g(x) \neq 0$

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry \neq 0 \quad , g(x) \neq 0$$

الحل العام $y = y_c + y_p$

The solution = Complementery function + Particuler Integlal

ايجاد y_c نفس الطرق السابقة

ايجاد y_p
1. The method of undetermined Coefficient (طريقة شبه مطولة)
2. by inverse (D) operator (طريقة تكاملات واشتقاقات)

① The method of undetermined cofficient

(A) g(x) is Constant

(B) g(x) is Polynomial

(C) g(x) is exponetial

(D) g(x) is (دالة cos sin)

A) $g(x)$ is constant

eq is written:

$$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=C \quad \text{---} (*)$$

Let $\quad y_p = c_1 \Rightarrow \frac{dy}{dx}P=0$

$$\frac{d^2y}{dx^2}P=0$$

Subsitute in eq (*)

$$0+0+Rc_1=C \quad , \quad c_1=y_p=\frac{C}{R}$$

ex: Solve $\quad y''-3y'-y=-2$

ans: $\quad m^2-3m-4=0 \Rightarrow (m-4)(m+1)=0$

$$\Rightarrow m_1=4, \; m_2=-1$$

$$y_c = c_1e^{4x}+c_2e^{-x}$$

$$y_p=\frac{C}{R}=\frac{-2}{-4}=\frac{1}{2}$$

$$y=y_c+y_p=c_1e^{4x}+c_2e^{-x}+\frac{1}{2}$$

ex: $3y'' - 6y' = 18$ --- *

ans: $3m^2 - 6m = 0$

$\Rightarrow m(3m-6) = 0 \Rightarrow m_1 = 0,\ m_2 = 2$

$y_c = c_1 e^{0x} + c_2 e^{2x}$

$y_c = c_1 - c_2 e^{2x}$

y_p: Let $y_p = c_3 x$

$y'_p = c_3 \Rightarrow y''_p = 0$

Substitute in eq *

$\Rightarrow 3 * 0 - 6 c_3 = 18$

$\Rightarrow c_3 = -3$

$\Rightarrow y_p = -3x$

$y = y_p + y_c$

$\Rightarrow y = c_1 + c_2 e^{2x} + (-3x)$

* إذا احد الجذور = 0 فان $y_p = cx$ ونعمل مشتق ونعوض .

(B) $g(x)$ is Polynomial (كثيرة حدود)

$a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$

$y_p = \alpha + \alpha_1 x + \alpha_2 x^2 + \alpha_n x^n$ (الغرض إيجاد $\alpha_0, \alpha_1, \alpha_2 \ldots$)

طريقة الحل مطولة لكن بسيطة وتحتاج إلى تركيز بالحل.

ex: Solve $y'' - 4y' + 4y = 4x + 8x^3$ —(*)

ans: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$, $m_1 = m_2 = 2$

$y_c = (c_1 + c_2 x)e^{2x}$

$y_p = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ —(1) توقفنا عند x^3 لأن أعلى أس في السؤال هو x^3

$y_p' = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2$ —(2)

$y_p'' = 2\alpha_2 + 6\alpha_3 x$ —(3)

Sub. eq. (1) (2) (3) in eq. (*)

$\Rightarrow 2\alpha_2 + 6x\alpha_3 - 4\alpha_1 + 8\alpha_2 x + 12\alpha_3 x^2 + 4\alpha_0$

$+ 4\alpha_1 x + 4\alpha_2 x^2 + 4\alpha_3 x^3 = 4x + 8x^3$

بعدها نعزل كل قوى كأنها حدود

$$(4\alpha_0 - 4\alpha_1 + 2\alpha_2) + (4\alpha_1 - 8\alpha_2 + 6\alpha_3)x$$

$$+ (4\alpha_2 - 12\alpha_3)x^2 + 4\alpha_3 \cdot x^3 = 4x + 8x^3$$

$$4\alpha_0 - 4\alpha_1 + 2\alpha_2 = 0 \quad *$$

(نستخرج القوى المتساوية من الطرفين)

$$4\alpha_1 - 8\alpha_2 + 6\alpha_3 = 4 \quad **$$

$$4\alpha_2 - 12\alpha_3 = 0 \quad ***$$

$$4\alpha_3 = 8 \quad ****$$

$$\Rightarrow \alpha_3 = 2 \text{ , sub in } ***$$

$$\Rightarrow \alpha_2 = 6 \text{ , sub } \alpha_2, \alpha_3 \text{ in } **$$

$$\Rightarrow \alpha_1 = 10$$

$$\text{Sub } \alpha_3, \alpha_2, \alpha_1 \text{ in } *$$

$$\Rightarrow \alpha_0 = 7$$

$$\therefore \quad y_p = 7 + 10x + 6x^2 + 2x^3$$

$$\Rightarrow y = y_c + y_p$$

$$= (c_1 + c_2 x)e^{2x} + 7 + 10x + 6x^2 + 2x^3$$

C) $g(x)$ is exponential Te^{rx}

if m_1 and $m_2 \neq r \Rightarrow y_p = \alpha e^{rx}$ وبعدها نشتقه

if m_1 or $m_2 = r \Rightarrow y_p = \alpha x e^{rx}$

if m_1 and $m_2 = r \Rightarrow y_p = \alpha x^2 e^{rx}$

وبعدها نشتق y_p' , y_p'' ونعوض ونستخرج α .

ex: solve $y'' + 2y' + y = e^{x}$

ans: sol. $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$\Rightarrow m_1 = m_2 = -1 \ , \ y_c = (c_1 + c_2 x) e^{-x}$

$\because r = 1 \neq m_1, m_2 \Rightarrow y_p = \alpha e^{x}$

$\Rightarrow y_p' = \alpha e^{x}$

$\Rightarrow y_p'' = \alpha e^{x}$,

$\Rightarrow \alpha e^{x} + 2\alpha e^{x} + \alpha e^{x} = e^{x}$

$4\alpha e^{x} = e^{x} \Rightarrow \alpha = \frac{1}{4}$

$\therefore y_p = \frac{1}{4} e^{x}$

$y = y_c + y_p \Rightarrow y = (c_1 + c_2 x) e^{-x} + \frac{1}{4} e^{x}$

Ex : Solve $(3D^2+10D-8)y=7e^{-4x}$ — *

يعني مشتقة أولى Dy الرمز D معناه مشتقة *

ثانية $= D^2y$

Sol: $3m^2+10m-8=0$

$(3m-2)(m+4)=0 \Rightarrow m_1=\frac{2}{3}, m_2=\frac{-4}{1}$

$y_c = c_1 e^{\frac{2}{3}x} + c_2 e^{-4x}$

Let : $y_p = \alpha x e^{-4x}$ — ① , because : $r=-4=m_2$

$y_p' = -4\alpha x e^{-4x} + \alpha e^{-4x}$ — ②

$y_p'' = 16\alpha x e^{-4x} - 4\alpha e^{-4x} - 4\alpha e^{-4x}$ — ③

Sub eq. ① ② ③ in eq. *

$48\alpha x e^{-4x} - 24\alpha e^{-4x} - 40\alpha x e^{-4x} + 10\alpha e^{-4x}$
$-8\alpha x e^{-4x} = 7e^{-4x}$

$\Rightarrow \quad -14\alpha e^{-4x} = 7e^{-4x}$

$\Rightarrow \quad \alpha = \frac{-7}{14} = \frac{-1}{2}$

$y = y_c + y_p$

$= c_1 e^{\frac{2}{3}x} + c_2 e^{-4x} - \frac{1}{2} x e^{-4x}$

D) $g(x)$ is $(F \sin nx + H \cos nx)$

or one of them.

$$y_p = L \sin nx + M \cos nx$$

$$y_p' = nL \cos nx - nM \sin nx$$

$$y_p'' = -n^2 L \sin nx - n^2 M \cos nx$$

* اذا كانت جذور y_c أعداد مركبة مثلاً $m = 2+i$ ،

$$y_p = x\,(C_3 \cos nx + C_4 \sin nx)$$

• if $n = 1$ ↘

$$y_p' = -C_3 x \sin x + C_3 \cos x + C_4 x \cos x$$
$$+ C_4 \sin x$$

$$y_p'' = -C_3 x \cos x - C_3 \sin x - C_3 \sin x$$
$$- C_4 x \sin x + C_4 \cos x + C_4 \cos x$$

$$\Rightarrow y_p'' = -(C_3 + x C_4) \sin x + C_4 \cos x$$
$$+ (C_4 - x C_3) \cos x - C_3 \sin x$$

نعوض y_p , y_p' , y_p'' بالمعادلة ونجد قيم الثوابت لإيجاد قيمة y_p .

ex: $y'' + y = \cos x$ ...(*), where $y(0) = 3$
$y'(0) = 0$

ans: $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m_1 = m_2 = \pm j$ (عدد مركب)

$y_c = C_1 \cos x + C_2 \sin x$

Let: $y_p = x(C_3 \cos x + C_4 \sin x)$

$y_p = C_3 x \cos x + C_4 x \sin x$ ①

$y_p' = -C_3 x \sin x + C_3 \cos x + C_4 x \cos x + C_4 \sin x$ ②

$y_p'' = -(C_3 + x C_4) \sin x + C_4 \cos x + (C_4 - x C_3) \cos x - C_3 \sin x$ ③

Sub eq. ① ② ③ in eq. (*)

$-2C_3 \sin x - x C_4 \sin x + 2C_4 \cos x - x C_3 \cos x + C_3 x \cos x + C_4 x \sin x = \cos x$

$\Rightarrow C_4 = \frac{1}{2}, \; C_3 = 0 \Rightarrow y_p = \frac{x}{2} \sin x$

$y = y_c + y_p = C_1 \cos x + C_2 \sin x + \frac{x}{2} \sin x$

$y(0) = 3 \Rightarrow 3 = C_1(1) + C_2(0) + 0 \Rightarrow C_1 = 3$

$y'(0) = 0 = C_1 \cos x - C_2 \sin x + \frac{x}{2} \cos x + \frac{1}{2} \sin x$

$\Rightarrow C_1 = 0 \quad \therefore \; y = 3 \cos x + \frac{x}{2} \sin x$

2) by Inverse (D) operator

D = differential operator (مشتقة)

$\frac{1}{D}$ = Integral operator (تكامل)

$$\frac{dy}{dx} = Dy \quad , \quad \frac{dy^2}{dx} = D^2 y$$

$$\int y \, dy = \frac{1}{D} y$$

* طريقة (D) operator أسهل وأقصر من الطريقة الاولى لذلك حاول فهمها .

(D) operator
- g(x) is exponential
- " " Polynomial
- " " Trigonometric function

وهناك أسئلة تشمل أكثر من نوع بنفس السؤال سنتطرق اليها الطريقة

a) $g(x)$ is exponential e^{px}

if m_1 and $m_2 \neq p \Rightarrow Y_p = \dfrac{1}{F(D)} e^{px} = \dfrac{1}{F(p)} e^{px}$

(أي نعوض المعادلة كاملة)

if m_1 or / and $m_2 = p \Rightarrow Y_p = \dfrac{1}{F(D)} e^{px} = \dfrac{1}{F(D+p)} e^{px}$

ex: Solve $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = e^{4x}$

sol: $(D^2 - D - 6)\, y = e^{4x}$

Y_c: $m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

$\Rightarrow m_1 = 3,\ m_2 = -2,\ Y_c = c_1 e^{3x} + c_2 e^{-2x}$

$p = 4 \neq m_1, m_2 \Rightarrow Y_p = \dfrac{1}{(D^2 - D - 6)} e^{4x}$

$Y_p = \dfrac{1}{(4^2) - (4) - (6)} e^{4x} \Rightarrow Y_p = \dfrac{1}{6} e^{4x}$

$Y = Y_c + Y_p$

$\Rightarrow Y = c_1 e^{3x} + c_2 e^{-2x} + \dfrac{1}{6} e^{4x}$

ex; $(D^2+8D+16)\,y = 6x\,e^{4x}$

$m^2-8m+16=0$

$(m-4)^2=0 \Rightarrow m_1=m_2=4$

$y_c = (C_1+C_2x)\,e^{4x}$

$\because\ P=4=m_1=m_2$

$\therefore\ y_p = \dfrac{1}{D^2-8D+16}\,6x\,e^{4x}$

$\Rightarrow y_p = \dfrac{1}{(D-4)^2}\,6x\,e^{4x}$

$\Rightarrow y_p = 6e^{4x}\,\dfrac{1}{(D-4+\underset{P}{4})^2}\,x$

$\Rightarrow y_p = 6e^{4x}\,\dfrac{1}{D^2}\,x$

(تكامل مرتين) نكامل x

$\Rightarrow y_p = 6e^{4x}\,\dfrac{1}{D}\left(\dfrac{x^2}{2}\right)$

$= 6e^{4x}\left(\dfrac{x^3}{6}\right)$

$= x^3e^{4x}$

$\therefore\ y = (C_1+C_2x)\,e^{4x}+x^3e^{4x}$

B) $g(x)$ is polynomial , $g(x) = x^n$

$$y_p = \frac{1}{F(D)} x^p = (a_0 + a_1 D + a_2 D^2 + \cdots a_p D^p) x^p$$

ex: Solve $(D^2 - D + 6)y = 4x^3 + 3x^2$

Sol; $m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

$\Rightarrow m_1 = +3 , m_2 = -2$

$$y_c = C_1 e^{3x} + C_2 e^{-2x}$$

للحل اس 3 ونتوقف لان الدالة اعلى قوة لها

$$y_p = \frac{1}{D^2 - D - 6} (4x^3 + 3x^2)$$

نقسمه قسمة طويلة لجعل D في البسط ونشتق الدالة.

$$\begin{array}{r|l} & -\frac{1}{6} + \frac{1}{36}D - \frac{7}{216}D^2 + \frac{13}{1296}D^3 \\ \hline -6 - D + D^2 & 1 \\ & +1 + \frac{1}{6}D - \frac{1}{6}D^2 \\ \hline & 0 - \frac{1}{6}D + \frac{1}{6}D^2 \\ & \frac{1}{6}D - \frac{1}{36}D^2 + \frac{1}{36}D^3 \\ \hline & \frac{7}{36}D^2 + \frac{1}{36}D^3 \\ & \frac{7}{36}D^2 + \frac{7}{216}D^3 - \frac{7}{216}D^4 \\ \hline & -\frac{13}{216}D^3 + \frac{7}{216}D^4 \\ & -\frac{13}{216}D^3 - \frac{13}{1296}D^4 + \frac{13}{1296}D^5 \end{array}$$

$$y_p = -\frac{1}{6} + \frac{1}{36}D - \frac{7}{216}D^2 + \frac{13}{1296}D^3 (4x^3 + 3x^2)$$

$$= -\frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x^2 + \frac{1}{6}x$$

$$-\frac{7}{216} D(12x^2) - \frac{7}{216} D(6x) + \frac{13}{1296} D$$

$$D(12x^2) + \frac{13}{1296} D[D(6x)]$$

$$y_p = -\frac{2}{3}x^3 - \frac{1}{6}x^2 + \frac{1}{6}x - \frac{7}{9}x - \frac{7}{36} + \frac{13}{54}$$

$$y_p = -\frac{2}{3}x^3 - \frac{1}{6}x^2 - \frac{11}{18}x + \frac{5}{108} \Rightarrow y_p = \frac{-72x^3 + 18x^2 + 66x - 5}{108}$$

$$y = y_c + y_p = C_1 e^{3x} + C_2 e^{-2x} + \left(\frac{-72x^3 + 18x^2 + 66x + 5}{108}\right)$$

c-) $g(x)$ is Trigonometric function.

$$g(x) = \sin px \quad \text{or} \quad \cos px$$

$$e^{ipx} = \cos px + i \sin px$$

$$y_p = \text{Real}\left[\frac{1}{F(D)} e^{ipx}\right], \text{ if } f(x) = \cos px$$

$$= \text{Real}\left[\frac{1}{F(ip)} e^{ipx}\right], \quad F(ip) \neq 0$$

$$y_p = \text{Im}\left[\frac{1}{F(ip)} e^{ipx}\right], \quad F(ip) \neq 0$$

ex: Solve and find y_p for:

$$y'' - y = \cos x$$

ans/ $y_c = c_1 e^{x} + c_2 e^{-x}$

$$y_p = \text{Real}\left[\frac{1}{D^2 - 1} e^{ix}\right]$$

$$= \text{Real}\left[\frac{1}{i^2 - 1} e^{ix}\right] = \text{Real}\left[\frac{1}{-1-1} e^{ix}\right]$$

$$= \text{Real}\left[\frac{1}{-2} e^{ix}\right] = -\frac{1}{2} \cos x$$

* إذا كانت الدالة كثيرة حدود أو مثلثية مع e^{px}

$$g(x) = f(x)\, e^{px}$$

where $f(x) = x^n$ or $\sin x$ or $\cos x$

$$y_p = \frac{1}{F(D)} f(x)\, e^{px} = e^{px} \frac{1}{F(D+P)} f(x)$$

ex: Solve $(D^2 - 4)y = e^{3x} \sin 2x$

Sol: $y_c = c_1 e^{2x} + c_2 e^{-2x}$

$$y_p = \frac{1}{D^2 - 4} e^{3x} \sin 2x = e^{3x} \frac{1}{(D+3)^2 - 4} \sin 2x$$

(نزحف D بمقدار الأس)

$$\Rightarrow y_p = e^{3x} \frac{1}{(D^2 - 6D + 9) - 4} \sin 2x$$

$$= e^{3x} \frac{1}{D^2 + 6D + 5} \sin 2x$$

$$\Rightarrow y_p = e^{3x}\, \mathrm{Im}\left[\frac{1}{D^2 + 6D + 5} e^{i2x}\right] = e^{3x}\, \mathrm{Im}\left[\frac{e^{i2x}}{(2i)^2 + 6(2i) + 5}\right]$$

$$\Rightarrow y_p = e^{3x}\, \mathrm{Im}\left[\frac{1 - 12i}{(1+12i)(1-12i)} e^{2ix}\right] = e^{3x}\, \mathrm{Im} \frac{1-12i}{1+144} e^{2ix}$$

$$y_p = \left\{\frac{1}{145}\, \mathrm{Im}\, e^{2ix} - \frac{12}{145}\, \mathrm{Real}\, e^{2ix}\right\} e^{3x}$$

$$y_p = e^{3x}\left[\frac{1}{145} \sin 2x + \frac{-12}{145} \cos 2x\right]$$

$$y = y_c + y_p$$

Euler function :-

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_1 y = 0 \quad —(1)$$

Let $x = e^t \Rightarrow t = \ln x \Rightarrow \frac{dt}{dx} = \frac{1}{x} \quad —(2)$

Sub in eq (1) The independed variable (t) :

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \quad —*$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt} \quad —**$$

(نعوض معادلات * و ** في السؤال ونحل بعد الترتيب بالقانون العام)

ex :- Solve $x^2 y'' + x y' - 4y = 0 \quad --*$

sol, $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

Sub in ~~eq~~ --*

$$x^2\left[\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right] + x\left(\frac{1}{x}\frac{dy}{dt}\right) - 4y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - 4y = 0$$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

$$y_c = C_1 e^{2t} + C_2 e^{-2t}$$

$$y_c = C e^{2\ln x} + C_2 e^{-2\ln x}$$

$$y_c = C_1 e^{\ln x^2} + C_2 e^{\ln x^2}$$

$$y_c = C_1 x^2 + \frac{C_2}{x^2}$$

Higher order liner D.E

ex: Solve $(D^4+3D^3+3D^2+D)y=0$

Sol: $m^4+3m^3+3m^2+m=0$

$\Rightarrow m(m^3+3m^2+3m+1)=0$

to solve $m^3+3m^2+3m+1=0$

Let $m=-1 \Rightarrow (-1)^3+3(-1)^2+3(-1)+1=0$

$$\begin{array}{r|l} & m^2+2m+1 \\ \hline m+1 & m^3+3m^2+3m+1 \\ & m^3+m^2 \\ \hline & 2m^2+3m+1 \\ & 2m^2+2m \\ \hline & m+1 \\ & m+1 \\ \hline & 0 \quad 0 \end{array}$$

$\Rightarrow (m+1)m(m^2+2m+1)=0$

$\Rightarrow m(m+1)(m+1)^2=0$

$m_1=0,\ m_2=-1,\ m_3=-1$

$m_4=-1$

$y=c_1+(c_2+c_3x+c_4x^2)e^{-x}$

Simultaneous liner D.E.

1- Elimination of the independent variable

2- Eliminating of one or more Dependent variable

(الطريقتين تكون طريقة حلهما عن طريق الحذف والتعويض

من ثم ايجاد كل احدى المعادلات المجهوله وتعويضها بالاخرى

1- Elimintion of the independent variable

If $\dfrac{dy}{dt} = f_1(x, y)$ --- (1

$\dfrac{dx}{dt} = f_2(x, y)$ --- (2

Divide eq (1 by eq (2, dt is elimination

$\Rightarrow \dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$ --- (3

eq. (3 is 1^{st} order D.E solved by Previon methods.

Ex :— Solve: $\frac{dy}{dt} = K_1 x - K_2 y$ --- (1

$\frac{dx}{dt} = -K_1 x$ --- (2

Sol :— Divide eq (1 by eq (2

$\Rightarrow \frac{dy}{dx} = -1 + K_2/K_1 \cdot \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = -1 + \frac{K_2}{K_1} \frac{y}{x}$ --- (3

eq (3 is homogeneous D.E

Let $y = Vx$, $\frac{dy}{dx} = V + x \frac{dV}{dx}$

$V + x \frac{dV}{dx} = -1 + \frac{K_2}{K_1} \cdot \frac{y}{x}$

$\Rightarrow V + x \frac{dV}{dx} = -1 + \frac{K_2}{K_1} V$

$\Rightarrow x \cdot \frac{dV}{dx} = -1 + \frac{K_2}{K_1} V - V$

$\Rightarrow x \frac{dV}{dx} = (\frac{K_2}{K_1} - 1) V - 1$

$\Rightarrow x \frac{dV}{dx} = (\frac{K_2 - K_1}{K_1}) V - 1$

$\Rightarrow \int \frac{dV}{(\frac{K_2 - K_1}{K_1}) V - 1} = \int \frac{dx}{x}$

$$\Rightarrow \frac{k_1}{k_2-k_1} Ln\left[\left(\frac{k_2-k_1}{k_1}\right)V-1\right] = Ln x + Ln C$$

$$\Rightarrow \left(\left(\frac{k_2-k_1}{k_1}\right)V-1\right)^{\wedge \frac{k_1}{k_2-k_1}} = x\,C$$

$$\Rightarrow \left(\frac{k_2-k_1}{k_1}\right)V-1 = C_1\, x^{\frac{k_2-k_1}{k_1}}$$

$$\Rightarrow V = \frac{k_1}{k_2-k_1}\left[C_1\, x^{\frac{k_2-k_1}{k_1}} + 1\right]$$

$$\Rightarrow \frac{y}{x} = \frac{k_1}{k_2-k_1}\left[C_1\, x^{\frac{k_2-k_1}{k_1}} + 1\right]$$

$$\Rightarrow y = x\,\frac{k_1}{k_2-k_1}\left[C_1\, x^{\frac{k_2-k_1}{k_1}} + 1\right] \quad \text{--- (4}$$

eq (2 $\frac{dx}{dt} = -k_1 x \Rightarrow \int \frac{dx}{x} = \int -k\, dt$

$\Rightarrow Ln\, x = -kt + C_2$

$\Rightarrow x = e^{-k_1 t}\, e^{C_2}$

$\Rightarrow x = C_3\, e^{-kt}$ — (5

Sub. eq (5 in eq (4

$$\Rightarrow y = C_3\, e^{-kt}\, \frac{k_1}{k_2-k_1}\left[C_1\, x^{\frac{k_2-k_1}{k_1}} + 1\right]$$

$$\Rightarrow y = C_3 e^{-k_1 t} \frac{k_1}{k_2-k_1}\left[C_1\left(C_3 e^{-k_1 t}\right)^{\frac{k_2-k_1}{k_1}} + 1\right]$$

$$\Rightarrow e^{-k_1 t} \frac{k_1}{k_2-k_1}\left[C_4 e^{-(k_2-k_1)t} + 1\right]$$

2) Eliminating of one or mor Dependent Variable.

ex: Solve

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y + \frac{d^2z}{dt^2} + 6\frac{dz}{dt} + 9z = 0 \quad (1)$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y + \frac{d^2z}{dt^2} - 3\frac{dz}{dt} + 2z = 0 \quad (2)$$

Sol; $(D^2 + D - 6)y + (D^2 + 6D + 9)z = 0 \quad (3)$

$(D^2 + 3D - 10) + (D^2 - 3D + 2)z = 0 \quad (4)$

eq (3) $\Rightarrow (D+3)(D-2)y + (D+3)^2 z = 0 \quad (5)$

eq (4) $\Rightarrow (D+5)(D-2)y + (D-2)(D-1)z = 0 \quad (6)$

Maltiply eq (5) by $(D+5)$ of multiply eq (6) by $(D+3)$.

$$(D+3)(D-2)(D+5)y+(D+3)^2(D+5)Z=0 \quad --(7)$$

$$(D+3)(D-2)(D+5)y+(D-2)(D-1)(D+3)Z=0 \quad (8)$$

from eq. (7) and (8)

$$\Rightarrow (D+5)(D+3)^2 Z-(D-3)(D-2)(D-1)Z=0$$

$$\Rightarrow (D+3)(D^2+8D+15-D^2+3D-2)Z=0$$

$$\Rightarrow (D+3)(11D+13)Z=0$$

$$\Rightarrow Z=C_1 e^{-3t}+C_2 e^{-\frac{13}{11}t} \quad --(9)$$

Substitute eq (9) in eq (3) or eq (4)

$$\Rightarrow (D^2+D-6)y+(D^2+6D+9)(C_1 e^{-3t}+C_2 e^{-\frac{13}{11}t})=0$$

$$\Rightarrow (D^2+D-6)y+D\left[\frac{-13}{11}C_1 e^{-\frac{13}{11}t}-3C_2 e^{-3t}-\frac{78}{11}C_1 e^{-\frac{13}{11}t}\right.$$

$$\left.-18C_2 e^{-3t}+9C_1 e^{-\frac{13}{11}t}+9C_2 e^{-3t}\right]=0$$

$$\Rightarrow (D^2+D-6)y+\frac{169}{121}C_1 e^{-\frac{13}{11}t}+9C_2 e^{-3t}$$

$$-\frac{78}{11}C_1 e^{-\frac{13}{11}t}-18C_2 e^{-3t}+9C_1 e^{-\frac{13}{11}t}+9C_2 e^{-3t}=0$$

$$\Rightarrow (D^2+D+6)y+\frac{400}{121}C_1 e^{-\frac{13}{11}t}=0$$

$$\Rightarrow (D^2+D-6)y=\frac{400}{121}C_1 e^{\frac{13}{11}t}=0$$

$$\Rightarrow y = \frac{1}{D^2+D-6}\left[\frac{-400}{121} c_1 e^{-\frac{13}{11}t}\right]$$

$$\Rightarrow y = \frac{1}{(\frac{13}{11})^2+(\frac{-13}{11})-6}\left[\frac{-400}{121} c_1 e^{-\frac{13}{11}t}\right]$$

$$\Rightarrow y_p = \frac{-121}{700} c_1 e^{-\frac{13}{11}t}$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_4 e^{-3t} + c_5 e^{2t} - \frac{121}{700} c_1 e^{-\frac{13}{11}t}$$

حل بعض أسئلة السنوات ، التي تضم أفكار مهمة

1) $(D^2-25)y = x^2 e^x - e^{5x}$

y_c . $m^2 - 25 = 0 \Rightarrow m_{1,2} = \pm 5$

$y_c = c_1 e^{5x} + c_2 e^{-5x}$

y_p :

لوجود دالتين في الطرف الآخر ولسهولة الحل سنقسم كل دالة على حدة :-

$y_p = y_{p_1} + y_{p_2}$

$$y_{p_1} = \frac{1}{D^2-25} x^2 e^x$$

$$y_{p_1} = e^x \frac{1}{(1+D)^2 + 25} (x^2)$$

$$y_{p_1} = \frac{1}{D^2+2D-24} x^2$$

$$\begin{array}{r|l} & -\frac{1}{24} - \frac{D}{288} - \frac{7D^2}{3956} \\ \hline -24+2D+D^2 & 1 \\ & 1 - \frac{1}{12}D - \frac{D^2}{24} \\ \hline & 0 + \frac{1}{12}D - \frac{D^2}{24} \\ & \frac{1}{12}D - \frac{D^2}{144} - \frac{D^3}{288} \\ \hline & \frac{7D^2}{144} + \frac{D^3}{288} \end{array}$$

$$y_{p_1} = e^x \left(\frac{-1}{24} - \frac{D}{288} - \frac{7D^2}{3456} \right)(x^2)$$

$$= e^x \left(-\frac{x^2}{24} - \frac{x}{144} - \frac{7}{1728} \right)$$

$$y_{p_2} = \frac{1}{D^2-25} \; -e^{5x}$$

$$y_{p_2} = \frac{1}{(D-5)(D+5)} \left(-e^{5x}\right)$$

$$y_{p_2} = \frac{1}{10\,(D-5)} \; -e^{5x} = \frac{1}{10(D+5-5)} \cdot -e^{5x}$$

$$y_{p_2} = -e^{5x}\,\frac{x}{10}$$

$$y = y_c + y_p$$

$$y = c_1 e^{5x} + c_2 e^{-5x} - \frac{e^{x} x^2}{24} - \frac{e^{x} x}{144} - \frac{7}{1728} e^{x} - \frac{e^{x} x}{10}$$

2) $y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$

y_c : $m^2 + 2m + 5 = 0$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1, \quad b = 2, \quad c = 5$

$m_{1,2} = \dfrac{-2 \pm \sqrt{2^2 - 4*1*5}}{2*1}$

$\Rightarrow m_{1,2} = -1 \pm 2i$

$\Rightarrow y_c = e^{-x}(c_1 \cos(2x) + c_2 \sin 2x)$

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = \dfrac{1.25e^{0.5x}}{D^2 - 2D + 5}$

$\Rightarrow y_{p_1} = \dfrac{1.25e^{0.5x}}{(0.5)^2 - 2(0.5) + 5} = 0.2e^{0.5x}$

$y_{p_2} = (a_0 \sin(4x) + a_1 \cos(4x))'' + 2[a_0 \sin(4x) + a_1 \cos(4x)]'$

$+ 5(a_0 \sin(4x) + a_1 \cos(4x))$

$= 40\cos(4x) - 55\sin 4x$

after Derivation and arrangement :

$$\left(-16 a_0 \sin(4x) - 16 a_1 \cos(4x)\right)$$
$$+ 2\left(a_0 \cos(4x) \times 4 - 4 a_1 \sin(4x)\right)$$
$$+ 5\left(a_0 \sin(4x) + a_1 \cos(4x)\right) =$$
$$40 \cos(4x) - 55 \sin(4x)$$

$$\Rightarrow -11 a_0 \sin(4x) + 8 a_0 \cos(4x) - 11 a_1 \cos(4x)$$
$$- 8 a_1 \sin(4x) = 40 \cos(4x) - 55 \sin(4x)$$

$$\Rightarrow (-11 a_0 - 8 a_1) \sin(4x) + (8a_0 - 11 a_1)$$
$$\cos(4x) = -55 \sin(4x) + 40 \cos(4x)$$

$$40 = 8 a_0 - 11 a_1 \quad \text{--- } (1)$$

$$-55 = -11 a_0 - 8 a_1 \quad \text{--- } (2)$$

from (1

$$\Rightarrow a_0 = \frac{40 + 11 a_1}{8} \quad \text{--- } (3)$$

sub (3 in (2

$$\Rightarrow -55 = -11\left(\frac{40 + 11 a_1}{8}\right) - 8 a_1$$

$$\Rightarrow a_1 = 0 \quad \text{sub in } (1$$

$$\Rightarrow a_0 = 5$$

$$\therefore y_{p_2} = 5\sin(4x) + 0\cdot\cos(4x)$$

$$\Rightarrow y_{p_2} = 5\sin(4x)$$

$$y_p = y_{p_1} + y_{p_2}$$

$$= 5\sin(4x) + 0.2e^{0.5x}$$

$$y = y_c + y_p$$

$$= e^{-x}\left(c_1\cos(2x) + c_2\sin(2x)\right) + 5\sin(4x) + 0.2e^{0.5x}$$

وجهة نظر

فهم السؤالين السابقين دليل على فهم أغلب الموضوع

Solutions of the differential equation by series

Solution of D-E By Series

-

Frobenius Method : This method is used to Solv. liner D-E with variable coff.

وحلها خطياً

طريقة الحل سوف تنقسم إلى 3 case بالإعتماد على قيم c_1, c_2

(خطوات الحل بشكل عام (الخطوات الأساسية) :-

1 - نكتب الصيغة

$$Y = a_0 X^{c} + a_1 X^{c+1} + a_2 X^{c+2} + \dots + a_{n-1} X^{c+n-1} + a_n X^{c+n}$$

2 - نشتق الصيغة

$$\frac{dy}{dx} = a_0 c x^{c-1} + a_1 (c+1) x^{c} + a_2 (c+2) x^{c+1} + \dots + a_{n-1} (c+n-1) x^{c+n-2} + a_n (c+n) x^{c+n-1}$$

3 - نشتق مشتقة الصيغة

$$\frac{d^2y}{dx^2} = a\, c (c-1) x^{c-2} + a_1 (c+1) c\, x^{c-1} + a_2 (c+2)(c+1) x^{c} + \dots + a_{n-1} (c+n-1)(c+n-2) x^{c+n-3} + a_n (c+n)(c+n-1) x^{c+n-2}$$

4 - نعوض y ، $\frac{dy}{dx}$ ، $\frac{d^2y}{dx^2}$ في المعادلة الأصلية (السؤال)

5 - بعد التعويض نسحب القيم المضروبة بأقل أس لل X والذي غالباً ما يكون (X^{c-1}) ونجد من خلال قيم C_1 و C_2

بعد إيجاد قيم C_1 , C_2 سنبدأ بإجراء الاختبارات لمعرفة نوع case

Solution of D.E by Series

Case 1 $\rightarrow$ $C_1 - C_2 =$ ليس عدد صحيح

$$\Downarrow$$

$$y = A\, y_1 |_{c=c_1} + B\, y_2 |_{c=c_2}$$

* قيم y_1 , y_2 سنجدها خلال حل الأمثلة
* A , B ثوابت

Case 2 $\rightarrow$ $C_1 = C_2$

$$\Downarrow$$

$$y = A\, y |_{c=c_1=c_2} + B \frac{dy}{dc} \Big|_{c=c_1=c_2}$$

* نجد قيمة y وبعدها نشتقها بالنسبة لـ C لإيجاد $\frac{dy}{dc}$

$C_2 > C_1$

Case 3 A $C_2 - C_1 =$ عدد صحيح موجب

where $j = C_2 - C_1$, $a_j = \infty$

$$\Downarrow$$

$$y = A\, y |_{c=c_2} + B \frac{d}{dc} \left[(c-c_1) y \right]_{c=c_1}$$

* نحول كل C إلى $C - C_1$ ونشتق y

$C_2 > C_1$

Case 3 B $C_2 - C_1 =$ عدد صحيح موجب

where $j = C_2 - C_1$, $a_j \neq \infty$

$$\Downarrow$$

$$y = A\, y_1 |_{c=c_1} + B\, y_2 |_{c=c_2}$$

مخطط يوضح قوانين Series

* سيتم التطرق للحل بمثال لكل case .

Case 1 :—

ex:— Solve $4x\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+y=0$ — (x)

Sol:— $y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}\cdots+a_{n-1}x^{c+n-1}$
$+a_nx^{c+n}$ — (1)

$$\frac{dy}{dx}=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}$$
$$\cdots a_{n-1}(c+n-1)x^{c+n-2}+a_n(c+n)x^{c+n-1} \quad (2)$$

$$\frac{d^2y}{dx^2}=a_0c(c-1)x^{c-2}+a_1(c+1)cx^{c-1}$$
$$a_2(c+2)(c+1)x^c\cdots+a_{n-1}$$
$$(c+n-1)(c+n-2)x^{c+n-3}+a_n$$
$$(c+n)(c+n-1)x^{c+n-2} \quad (3)$$

Sub. (1) (2) (3) in (x)

$$4a_0c(c-1)x^{c-1}+4a_1c(c+1)x^c+4a_2(c+2)$$
$$(c+1)x^{c+1}+\cdots+a_{n-1}(c+n-1)(c+n-2)x^{c+n-2}$$
$$+a_n(c+n)(c+n-1)x^{c+n-1}+6a_0cx^{c-1}+$$
$$6a_1(c+1)x^c+6a_2(c+2)x^{c+1}+\cdots+6a_{n-1}(c+n-1)$$
$$x^{c+n-2}+a_n(c+n)x^{c+n-1}+a_0x^c+a_1x^{c+1}+a_2x^{c+2}$$
$$\cdots+a_{n-1}x^{c+n-1}+a_n\cdot x^{c+n}=0 \quad (4)$$

Equatin The coefficient of the Lowest power of (x) which is (x^{c-1}) in eq (4)

$4 \cdot a_0 c(c-1) + 6 a_0 c = 0$ --- (5)

Divide by $2a_0$

$2c(c-1) + 3c = 0 \Rightarrow c[2(c-1)+3] = 0$

$\Rightarrow c[2c+1] = 0 \Rightarrow c_2 = 0 , c_1 = -\frac{1}{2}$

$c_1 - c_2 = 0 - (-\frac{1}{2}) = \frac{1}{2}$ كسر ليس عدد صحيح

$y = A y_1 |_{c=c_1} + B y_2 |_{c=c_2}$

Equaling the coefficiente of the x^{c+n} or x^{c+n-1} between a_{n-1} or a_{n-2} or a_{n-3} ... with (a_n)

Equaling The coefficients of x^{c+n-1} in eq (4)

$4 a_n (c+n)(c+n-1) + 6 a_n (c+n) + a_{n-1} = 0$

$\Rightarrow 4 a_n (c+n)(c+n-1) + 6 a_n (c+n) = -a_{n-1}$

$\Rightarrow a_n (c+n)[4(c+n-1)+6] = -a_{n-1}$

$\Rightarrow \frac{a_n}{a_{n-1}} = \frac{1}{(c+n)[4(c+n-1)+6]}$ (تسمى الصيغة المتعاقبة)

when $c=0$

$\Rightarrow \frac{a_n}{a_{n-1}} = -\frac{1}{n[4(n-1)+6]} = -\frac{1}{n(4n-4+6)}$

$$\frac{a_n}{a_{n-1}} = \frac{1}{n(4n+2)} = -\frac{1}{2n(2n+1)}$$

$n=1 \Rightarrow \frac{a_1}{a_0} = \frac{-1}{2*1(2*1+1)} = \frac{-1}{6} = \frac{-1}{3!} \Rightarrow a_1 = \frac{1}{3!} a_0$

$n=2 \Rightarrow$

$$\frac{a_n}{a_{n-1}} = \frac{a_2}{a_1} = -\frac{1}{2*(2(2*2+1))} = \frac{-1}{20}$$

$$\frac{a_2}{a_0} = \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} = -\frac{1}{6} * \frac{-1}{20} = \frac{1}{120} = \frac{1}{5!}$$

$\therefore a_2 = \frac{1}{5!} a_0$, sub in eq (1)

$$y = a_0 - \frac{1}{3!} a_0 x + \frac{1}{5!} a_0 x^2 - \frac{1}{7!} a_0 x^3 \dots$$

$$y_1 = a_0 \left[1 - \frac{1}{3!} x + \frac{1}{5!} x^2 - \frac{1}{7!} x^3 \dots \right]$$

for $c = -\frac{1}{2}$

$$\frac{a_n}{a_{n-1}} = -\frac{1}{(n-\frac{1}{2})\left[4(-\frac{1}{2}+n-1)\right]+6}$$

$$= \frac{-1}{\frac{2n-1}{2}\left[4\left(\frac{-1+2n-2}{2}\right)+6\right]}$$

$$= \frac{-1}{\frac{2n-1}{2}\left[\frac{-4+8n-8+12}{2}\right]} = \frac{-1}{\frac{2n-1}{2}\left(\frac{8n}{2}\right)}$$

$$\frac{a_n}{a_{n-1}} = \frac{-1}{4n^2-2n} = \frac{-1}{2n(2n-1)}$$

$$n=1 \Rightarrow \frac{a_1}{a_0} = \frac{-1}{2!} \Rightarrow a_1 = \frac{-1}{2} a_0$$

$$n=2 \Rightarrow \frac{a_2}{a_1} = \frac{-1}{4*3}$$

$$\frac{a_2}{a_0} = \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} = \frac{-1}{3*4}\left(\frac{-1}{2}\right) = \frac{1}{24} = \frac{1}{4!}$$

$$\Rightarrow a_2 = \frac{1}{4!} a_0$$

$$y_2 = a_0 x^{-\frac{1}{2}} - \frac{1}{2!} a_0 x^{-\frac{1}{2}+1} + \frac{1}{4} a_0 x^{-\frac{1}{2}+2} - \frac{1}{6} a_0 x^{-\frac{1}{2}+3}$$

$$y_2 = a_0 x^{-\frac{1}{2}} \left[1 - \frac{1}{2!} x + \frac{1}{4!} x^2 - \frac{1}{6!} x^3 + \cdots\right]$$

$$\Downarrow$$

$$y_2 = B x^{-\frac{1}{2}} \left[1 - \frac{1}{2!} x + \frac{1}{4!} x^2 - \frac{1}{6!} x^3 + \cdots\right]$$

$$y = y_1 + y_2$$

Case 2 : Roots of indicial Equation are Equal , IF $C_1 = C_2$

The general Equation : $y = A\, y|_{c=c_1=c_2} + B \left.\frac{dy}{dc}\right|_{c=c_{1,2}}$

Ex:

Solve : $x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} - y = 0$ — ⊛

Sol: ① نجد y و $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ (نفس المثال السابق)

② نعوض القيم عن y , $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ في المعادلة ⊛

$a_0 c(c-1)x^{c-1} + a_1 c(c+1)x^{c} + a_2 (c+2)(c+1) x^{c+1} + \dots + a_{n-1}(c+n-1)(c+n-2)x^{c+n-2} + a_n (c+n)(c+n-1) x^{c+n-1} + a_0 c x^{c-1} + a_1 (c+1) x^{c} + a_2 (c+2) x^{c+1} + \dots + a_{n-1}(c+n-1) x^{c+n-2} + a_n (c+n) x^{c+n-1} - a_0 c x^{c} + a_1 (c+1) x^{c+1} - a_2 (c+2) x^{c+2} - \dots - a_{n-1}(c+n-1) x^{c+n-1} - a_n (c+n) x^{c+n} - a_0 x^{c} - a_1 x^{c+1} - a_2 x^{c+2} - \dots - a_{n-1} x^{c+n-1} - a_n x^{c+n} = 0$

Equation the Cofficient of The Lowest Power (x)

$a_0 c(c-1) + a_0 c = 0 \quad \div a_0$

$c(c-1) + c = 0 \Rightarrow [c(c-1+1)] = 0$

$\Rightarrow c_1 = c_2 = 0$

Equation Coefficient of x^{c+n-1}

$a_n(c+n)(c+n-1) + a_n(c+n) - a_{n-1}(c+n-1)$

$- a_{n-1} = 0$

$\Rightarrow a_n(c+n)(c+n-1) + a_n(c+n) = a_{n-1}(c+n-1+1)$

Recurrance formula (العلاقة التكرارية)

$$\Rightarrow \frac{a_n}{a_{n-1}} = \frac{(\cancel{c+n})}{(\cancel{c+n})(c+n-\cancel{1}+\cancel{1})} = \frac{1}{c+n}$$

for $c=0$

$n=1 \Rightarrow \frac{a_1}{a_0} = \frac{1}{1} \Rightarrow a_1 = 1!\, a_0$

$n=2 \Rightarrow \frac{a_2}{a_1} = \frac{1}{2!}$

$$\frac{a_2}{a_0} = \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} = \frac{1}{1!} \times \frac{1}{2!} = \frac{1}{2!}$$

$n=3 \Rightarrow \frac{a_3}{a_2} = \frac{1}{3}$

$$\frac{a_3}{a_0} = \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \cdot \frac{a_1}{a_0}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{3!}$$

Sub in y

$$y_1 = a_0 x^0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 \ldots$$

$$y_1 = a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right]$$

$$y_1 = A \quad e^x$$

تذكر

$$e^x = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \right]$$

$$y_2 = \frac{dy}{dc}\bigg|_{c=c} = a_0\Big[x^c \ln x + \frac{1}{c+1}x^{c+1}\ln x - \frac{1}{(c+1)^2}x^{c+1} + \frac{1}{c+2}\Big\{\frac{1}{c+1}x^{c+2}\ln x - \frac{1}{(c+1)^2}x^{c+2}\Big] - \frac{1}{c+1}x^{c+2} + \frac{1}{(c+2)^2}$$

$$\Rightarrow \frac{dy}{dc} = a_0 \ln x\Big[x^c + \frac{1}{c+1}x^{c+1} + \frac{1}{(c+2)}\frac{1}{(c+1)}\cdot x^{c+2} \cdots\Big] - a_0\Big[\frac{1}{(c+1)}x^{c+1} + \frac{1}{(c+1)^2(c+2)} + \frac{1}{(c+1)(c+2)^2}x^{c+2} + \cdots\Big]$$

when $c=0$

$$y_2 = \frac{dy}{dc}\bigg|_{c=0} = a_0 \ln x\Big[1 + x + \frac{1}{2!}x^2 + \cdots\Big] - a_0\Big[x + \Big(\frac{1}{2}+\frac{1}{4}\Big)x^2 + \Big(\frac{1}{6}+\frac{1}{12}+\frac{1}{1}\Big)x^3 \cdots\Big]$$

$$\Rightarrow y_2 = a_0 \ln x - a_0\Big[x + \frac{3}{4}x^2 + \frac{11}{36}x^3 + \cdots\Big]$$

$$y = y_1 + y_2 = A e^x + B e^x \ln x - B\Big[x + \frac{3}{4}x^2 + \frac{11}{36}x^3 \cdots\Big]$$

ملاحظة مهمة :- $X^c = X^c \ln x$ الدالة
المشتقة

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$ و $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$ (وفرق بالإشارة)

Case 3 A: Root of Indicial Equation are Different by an Integer

if $c_2 > c_1$, $c_2 - c_1 =$ positive integer

$a_j \big|_{c=c_1} = \infty$, $j = c_2 - c_1$

The solution: $y = A\, y \big|_{c=c_2} + B \left. \dfrac{\partial\, [(c-c_1)\, y]}{\partial c} \right|_{c=c_1}$ — (*)

ex: Solve $x^2 y'' + (x^2 - 2x)\, y' + 2y = 0$

Sol//

Indicial Eq. $(c^2 - 3c + 2)\, a_0 = 0$

* هنا غير نبدأ الحل من Inditial eq. و ايجاد قيم C_2 و C_1

Sol // $c^2 - 3c + 2 = 0$

$\Rightarrow c_2 = 2, \quad c_1 = 1$

$c_2 > c_1, \quad c_2 - c_1 = 2 - 1 = 1$ positive integer

Recurrence eq., $a_n = \dfrac{-1}{c+n-2}\, a_{n-1}$

$a_j = a_{c_2 - c_1} = a_{2-1} = a_1$

$a_1 \big|_{c=c_1=1} = \dfrac{-1}{1+1-2}\, a_0$

$a_j = \dfrac{1}{0} = \infty$

The solution is according to equation (*):

- According to recurrence formula..

$$a_1 = \frac{-1}{c-1} a_0$$

$$a_2 = \frac{-1}{c} a_1 = \frac{-1}{c}\left(\frac{-1}{c-1} a_0\right)$$

$$a = \frac{1}{c(c-1)a_0} \quad , \quad a_3 = \frac{-1}{c(c-1)(c+1)} a_0$$

$$y = a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} + a_3 x^{c+3} \cdots$$

$$y = a_0 \left[x^c - \frac{1}{c-1} x^{c+1} + \frac{1}{(c-1)c} x^{c+2} - \frac{1}{(c-1)c(c+1)} x^{c+3} \right.$$

$$y_1 |_{c=c_2=2} = a_0 \left[x^2 - \frac{1}{1} x^3 + \frac{1}{2\times1} x^4 - \frac{1}{3\times2\times1} x^5 + \cdots \right]$$

$$y_1 = a_0 x^2 \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots \right]$$

$$y_1 = A x^2 e^{-x}$$

$$(c + c_1) y = (c-1) y = a_0 \left[(c-1) x^c - x^{c+1} + \frac{1}{c} x^{c+2} - \frac{1}{c(c+1)} x^{c+3} + \cdots \right]$$

$$\frac{\partial (c-c_1) y}{\partial c} = a_0 \left[(c-1) x^c \ln x + x^c - x^{c+1} \ln x + \frac{1}{c} x^{c+2} \ln x + \frac{1}{c(c+1)^2} x^{c+3} + \frac{1}{c^2(c+1)} x^{c+3} + \cdots \right]$$

$$\frac{\partial (c-c_1)y}{\partial c} = a_0 \ln x \left[c(c-1) x^{c} - x^{c+1} + \frac{1}{c} x^{c+2} - \frac{1}{c(c+1)} x^{c+3} \cdots \right] + a_0 \left[x^{c} - \frac{1}{c^2} x^{c+2} + \frac{1}{c(c+1)^2} x^{c+3} + \frac{1}{c^2(c+1)} x^{c+3} + \cdots \right]$$

$$\frac{\partial (c-c_1)y}{\partial c} \Big|_{c=c_1=1} = a_0 \ln x \left[-x^2 + x^3 - \frac{1}{2} x^4 + \cdots \right] + a_0 \left[x - x^3 + \frac{1}{4} x^4 + \cdots \right]$$

$$= -a_0 x^2 \ln x \left[1 - x + \frac{1}{2!} x^3 \cdots \right] + a x \left[1 - x^2 + \frac{3}{4} x^3 - \cdots \right]$$

$$= -a_0 x^2 e^{-x} \ln x + a_0 x \left[1 - x^2 + \frac{3}{4} x^3 \cdots \right]$$

$$y = A x^2 e^{-x} - B x^2 e^{-x} \ln x + B x \left[1 - x^2 + \frac{3}{4} x^3 \cdots \right]$$

$$y = (A - B \ln x) x^2 e^{-x} + B x \left[1 - x^2 + \frac{3}{4} x^3 + \cdots \right]$$

Case 3 B : If $C_2 > C_1$, $C_2 - C_1$ = Positive Integer

and $a_i|_{c=c_1} \neq \infty$ where $i = C_2 - C_1$

The Solution: $y = Ay_1|_{c=c_1} + By_2|_{c=c_2}$

The procedure is similler to Case 1

ex: Solve: $x^2y'' + (x - x^2)y' - y = 0$

Sol: Indicial eq. $ac(c-1) + a_0c - a_0 = 0$

$(c+1)(c-1) = 0 \Rightarrow c_1 = -1$, $c_2 = 1$

$a_i = a_{c_2 - c_1} = a_{1-(-1)} = a_2$

$a_2|_{c=c_1} = a_2|_{c=-1} = \frac{1}{2-1+1} = \frac{1}{2} \neq \infty$

Solution $y = Ay|_{c=c_1} + By|_{c=c_2}$

for $c_1 = -1$

$\frac{a_n}{a_{n-1}} = \frac{1}{n} \Rightarrow \frac{a_1}{a_0} = \frac{1}{1}$

$\frac{a_2}{a_1} = \frac{1}{2} \Rightarrow a_2 = \frac{1}{2} a_1$

$\frac{a_3}{a_2} = \frac{1}{3} \Rightarrow a_3 = \frac{1}{3} a_2 \Rightarrow a_3 \frac{1}{3*2} a_0$

$y_1 = a_0 x^{-1} + a_1 x^0 + a_2 x + a_3 x^2 \ldots$

$= a_0 x^{-1} \{ 1 - x + \frac{1}{2} x^2 + \frac{1}{3*2} x^3 \ldots \}$

$y = Ax^{-1} e^x$

for $c_2 = 1$

$a_n = \frac{1}{n} a_{n-1} \Rightarrow a_n = \frac{1}{n+2} a_{n-1}$

$a_1 = \frac{1}{1+2} a_0 \Rightarrow a_1 = \frac{1}{3} a_0$

$a_2 = \frac{1}{2+2} a_1 \Rightarrow a_2 = \frac{1}{4 \times 3} a_0$

$a_3 = \frac{1}{3+2} a_2 \Rightarrow a_3 = \frac{1}{5} a_2 \Rightarrow a_3 = \frac{1}{5 \times 4 \times 3} a_0$

$y_2 = a_0 [x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots]$

$= a_0 [x + \frac{1}{3} x^2 + \frac{1}{4 \times 3} x^3 + \frac{1}{5 \times 4 \times 3} x^4 + \cdots]$

$= a_0 x [1 + \frac{1}{3} x + \frac{1}{4 \times 3} x^2 + \frac{1}{60} x^3 + \cdots]$

$= B x [1 + \frac{1}{3} x + \frac{1}{2} x^2 + \frac{1}{60} x^3 + \cdots]$

$y = y_1 + y_2$

$= A x^{-1} e^{-x} + B x [1 + \frac{1}{3} x + \frac{1}{12} x^2 + \frac{1}{60} x^3 \cdots]$

Bessel's Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - k^2)y = 0 \quad (1)$$

Equation (1) is Bessel's equation of order (k)

General Solution :-

$$y = A J_k(x) + B J_{-k}(x)$$

if (k) is not an integer or Zero

where , $J_k(x)$: is Bessel's ~~Eq~~ function of first kind

$$y = A J_k(x) + B Y_k(x)$$

if (k) is an integer or Zero

where $Y_k(x)$: is Bessel's ~~Fun~~ function of 2^{nd} kind

J_k ، J_{-k} ، Y_k تعطى بشكل جداول *

عكس الاشارة

Modefial Bessel's Equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + K^2) y = 0$$

General solution :-

$$y = A\, I_k(x) + B\, I_{-k}(x)$$

if (K) is not an Integer or Zero

where: I_K : is modefial Bessel's function of 1st kind

$$y = A\, I_k(x) + B\, K_k(x)$$

if (K) is an Integer or Zero

where: K_k - Modifial Bessel's function of 2nd Kind.

تعطى بشكل جداول K_k , I_{-k} , I_k *

Proparties:

1) $k=0$ $\qquad J_0(0) = I_0(0) = 1$

2) $k>0$ $\qquad J_k(0) = I_k(0) = 0$

integral

$$Y_k(0) = K_k(0) = \infty$$

non integer — $J_k(0) = \pm I_k(0) = \pm\infty$

3) $$x\frac{d}{dx} I_k(\alpha x) = k I_k(\alpha x) + \alpha x I_{k+1}(\alpha x).$$

$$x\frac{d}{dx} K_k(\alpha x) = k K_k(\alpha x) - \alpha x K_{k+1}(\alpha x)$$

4) $$J_{-k}(\alpha x) = (-1)^k \cdot J_k(\alpha x)$$
$$Y_{-k}(\alpha x) = (-1)^k \cdot Y_k(\alpha x)$$

Bessel's equation

$$I_{-k}(\alpha x) = I_k(\alpha x)$$
$$K_{-k}(\alpha x) = K_k(\alpha x)$$

Modified B.E.

ex: Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2 y = 0$ —— (*)

with Boundery Conditions:

$x = 0.0195 \quad , \quad y = 105$

$x = 0.078 \qquad \frac{dy}{dx} = 0.0195\, y$

Sol/ Eq. * is modifed Bessel's equation of order 0

$y = A I_0(x) + B K_0(x)$

$x = 0.0195 \quad , \quad y = 105$

$\Rightarrow y = A I_0(0.0195) + B K_0(0.0195) = 105$

From Tables: $I_0(0.0195) = 1$, $K_0(0.0195) = 4.0285$

$\Rightarrow A(1) + B(4.0285) = 105$ —— ①

* $\frac{d}{dx} I_0(\alpha x) = \alpha x I_1(\alpha x) \qquad \alpha = 1$

$\frac{d}{dx} I_0(x) = I_1(x)$

$\frac{x\,d}{dx} K_0(x) = -x K_1(x)$

$\frac{d}{dx} K_0(x) = -K_1(x)$

$\frac{dy}{dx} = B\frac{dK_0}{dx} + A\frac{dI_0}{dx}$

$$\Rightarrow A I_1(0.078) - B K_1(0.078) = -(0.0195)$$

$$[A I_0(0.0195) + B K_0(0.0195)]$$

from Tables:

$$I_1(0.078) = 0.04$$

$$K_1(0.078) = 13.3742$$

$$\Rightarrow 0.04A - 13.374B = -0.0195(105) \quad (2)$$

from eq (1) and (2)

$$\Rightarrow A = 102.4$$

$$B = 0.657$$

$$y = 102.4 I_0(x) + 0.652 K_0(x).$$

* المطلوب إيجاد في معادلة بيسل هو قيم A ، B

* المعادلة مهمة جداً في الكورس الثاني وخصوصاً في موضوع الـ Modeling (المودبل الرياضي)

* يتم تحديد نوع المعادلة من إشارة K او اشارة / x حبد في المعادلات ، ومنه يتم إستخدام القوانين للوصول إلى حل السؤال بإيجاد قيم A ، B .

Laplace transforms

Laplace Transformation

It is a mathematic method for solving D.E.

If $F(t)$ is a continuous function of an independent variable (t) for all values of $t > 0$

if $\mathcal{L}\, F(t) = f(s) = \int_0^\infty F(t)\, e^{-st}\, dt$

where s = variable (s operation)

* ملاحظات :-

1- لابلاس ولابلاس انفيرس أشبه بالتكامل والمشتقة ولكن بقوانين وصيغة مختلفة ،

2- عند حل أي سؤال يتعلق بلابلاس (تحويلات) أولى الخطوات هي (الاعتماد على القوانين) أذا لم نجد ذلك (نعتمد على التكاملات الجزئية بعدها القوانين)، وأذا لم نجد حل (نستخدم تحويلات خاصة سنتطرق اليها وبعدها تكاملات جزئية وبعدها القوانين العامة .) أي هنالك ثلاث طرق للحل .

3- الرمز $\mathcal{L}$ ← laplace Transform

$\mathcal{L}^{-1}$ ← inverse laplace Transform

Basic Transforms :-

القوانين العامة :

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^n (n=0,1,..)	$\frac{n!}{s^{n+1}}$
4	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	Cos wt	$\frac{s}{s^2+w^2}$
7	Sin wt	$\frac{w}{s^2+w^2}$
8	Cosh at	$\frac{s}{s^2-a^2}$
9	Sinh at	$\frac{a}{s^2-a^2}$
10	$e^{at}\cos wt$	$\frac{s-a}{(s-a)^2+w^2}$
11	$e^{at}\sin wt$	$\frac{w}{(s-a)^2+w^2}$

examples

1) $\mathcal{L}\, t^5 = \dfrac{5!}{s^6} = \dfrac{5*4*3*2*1}{s^6} = \dfrac{120}{s^6}$

2) $\mathcal{L}\, \sin 6t \cdot \sin 4t$

$= \mathcal{L}\, \frac{1}{2}\left(\cos 2t - \cos 10t\right)$

$= \frac{1}{2}\mathcal{L}\, \cos 2t - \frac{1}{2}\mathcal{L}\, \cos 10t$

$= \frac{1}{2}\left(\dfrac{s}{s^2+4} - \dfrac{s}{s^2+100}\right)$

$= \dfrac{48s}{(s^2+4)(s^2+100)}$

3) $\mathcal{L}\, \cos^3 2t = \mathcal{L}\, \frac{1}{4}\left(3\cos 2t + \cos 6t\right)$

$= \frac{3}{4}\mathcal{L}\, (\cos 2t + \frac{1}{4}\cos 6t)$

$= \frac{3}{4}\,\dfrac{s}{s^2+4} + \frac{1}{4}\,\dfrac{s}{s^2+36}$

$= \dfrac{s(s^2+28)}{(s^2+4)(s^2+36)}$

3) $\mathcal{L}\sinh^3 2t = \mathcal{L}\left(\frac{e^{2t}-e^{-2t}}{2}\right)^3$

$= \mathcal{L}\,\frac{1}{8}\left(e^{8t} - 3e^{2t} + 3e^{-2t} - e^{-8t}\right)$

$= \frac{1}{8}\mathcal{L}\, e^{8t} - \frac{3}{8}\mathcal{L}\, e^{2t} + \frac{3}{8}\mathcal{L}\, e^{-2t} - \frac{1}{8}\mathcal{L}\, e^{-8t}$

$= \frac{1}{8}\cdot\frac{1}{s-6} - \frac{3}{8}\cdot\frac{1}{s-2} + \frac{3}{8}\cdot\frac{1}{s+2} - \frac{1}{8}\cdot\frac{1}{s+6}$

$= \frac{1}{8}\left[\frac{1}{(s-6)} - \frac{1}{(s+6)}\right] - \frac{3}{8}\cdot\left[\frac{1}{s-2} + \frac{1}{s+2}\right]$

$= \frac{1}{8}\left[\frac{12}{s^2-36} - \frac{12}{s^2-4}\right. = \frac{48}{(s^2-36)(s^2-4)}$

* يجب مراجعة تحويلات الدوال المثلثية و منهج الرياضيات
المرحلة الأولى / الكورس الثاني.

laplace Transform of Derivatives :-

* مجموعة من أسئلة حل المعادلة التفاضلية باستخدام تحويلات لابلاس

$$\mathcal{L}\frac{df(t)}{dt} = \mathcal{L}\, y'(t) = \int_0^{\infty} \frac{d(f(t))}{dt}\, e^{-st}\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$$dv = \frac{df(t)}{dt}, \quad u = e^{-st}$$

$$v = f(t), \quad du = -s\,e^{-st}$$

$$\mathcal{L}\frac{df(t)}{dt} e^{-st} = f(t)\, e^{-s(t)}\Big|_0^{\infty} - \int_0^{\infty} f(t)\,(-s\,e^{-st})$$

$$\mathcal{L}\frac{df(t)}{dt} = sF(s) - f(0) \quad \text{--- (1)}$$ حفظ

$$\mathcal{L}\frac{d^2F(t)}{dt^2} = s^2F(s) - sF(0) - f'(0) \quad \text{--- (2)}$$ حفظ

$$\mathcal{L}\frac{d^3F(t)}{dt^3} = s^3 f(s) - s^2 f(0) - s f'(0) - f''(0) \quad \text{(3)}$$ حفظ

in general :-

$$\mathcal{L}\frac{d^n f(t)}{dt^n} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s\, f^{n-2}(0) - f^{(n-1)}(0) \quad \text{--- (4)}$$

Shifting Theorem :-

$$\mathcal{L}\, e^{-\alpha t} F(t) = f(s+\alpha) \quad , \alpha = \text{constant}$$

Ex:- Find $\mathcal{L}\, e^{-\alpha t} \cos \beta t$

where α, β constant

Sol: let $F(t) = \cos \beta t$

$$f(s) = \frac{s}{s^2+\beta^2}$$

$$\mathcal{L}\, e^{-\alpha t} \cos \beta t = \frac{s+\alpha}{(s^2+\alpha^2)+\beta^2}$$

Inverse laplace Transform

$\mathcal{L}^{-1} F(s) = F(t)$

1- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

2- $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

3- $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+a}}\right\} = \frac{t^n}{\Gamma(n+1)} = \frac{t^n}{n!}$ *, if n is a positive integer.*

4- $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

5- $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

6- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

7- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

8- $\mathcal{L}^{-1}\left\{\frac{1}{(s+b)n+1}\right\} = \frac{e^{-bt}.t^n}{\Gamma(n+1)} = \frac{e^{-bt}.t^n}{n!}$ *, if n is a positive integer.*

9- $\mathcal{L}^{-1}\left\{\frac{s+b}{(s+b)^2+a^2}\right\} = e^{-bt}.\cos at$

10- $\mathcal{L}^{-1}\left\{\frac{1}{(s+b)^2+a^2}\right\} = \frac{1}{a}e^{-bt}.\sin at$

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ex: $\mathcal{L}^{-1}\left(\frac{s+8}{s^2+4s+5}\right)$ - solving

sol:

$$\mathcal{L}^{-1}\frac{s+8}{s^2+4s+5} = \mathcal{L}^{-1}\frac{(s+2)+6}{(s+2)^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\frac{(s+2)+6}{(s+2)^2+1} = \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1} + \frac{6}{(s+2)^2+1}\right)$$

$$= e^{-2t}\cos t + 6e^{-2t}\sin t$$

$$= e^{-2t}(\cos t + 6\sin t)$$

$$\mathcal{L}^{-1}\left(\frac{s+b}{(s+b)^2+a^2}\right) = e^{-bt}\cos at$$

$$\mathcal{L}^{-1}\left(\frac{a}{(s+b)^2+a^2}\right) = \frac{1}{a}e^{-bt}\sin at$$

* Inverting by shifting Theorem

$$\mathcal{L}^{-1} F(s+\alpha) = e^{-\alpha t} f(t)$$

ex: Find $\mathcal{L}^{-1} \dfrac{s}{s^2+2s+5}$

Sol: $\mathcal{L}^{-1} \dfrac{s}{s^2+2s+5} = \mathcal{L}^{-1} \dfrac{s}{s^2-2s+1+4}$

$$= \mathcal{L}^{-1} \frac{s}{(s-1)^2+4} = \mathcal{L}^{-1} \frac{s}{(s-1)^2+2^2}$$

$$F(s+1) = \frac{s+1}{s^2+2^2} = \frac{s}{s^2+2^2} + \frac{1}{s^2+2^2}$$

$$\mathcal{L}^{-1} F(s+\alpha) = e^{-\alpha t} f(t)$$

$$\Rightarrow \mathcal{L}^{-1} F(s+1) = e^{-t} f(t)$$

$$= \cos 2t + \frac{1}{2} \sin 2t$$

$$\Rightarrow f(t) = e^{t} \left\{ \cos 2t + \frac{1}{2} \sin 2t \right\}$$

Inverting by partial fraction:

If $F(s) = \frac{\theta(s)}{\phi(s)}$, where $\theta(s)$ an $\phi(s)$ are Polynomials is (s) and $\phi(s)$ is of higher degree than $\theta(s)$, so $F(s)$ can be expanded to its Partial Fraction

(إذا لم تصل لحل بعد التبسيط والعمل على القوانين العامة ، استخدم طريقة الكسور الجزئية)
ومن خلالها راح يتبسط السؤال وتكدر تحل بالقوانين العامة (الرئيسية) .

* طريقة حل الكسور الجزئية موجودة في منهج الرياضيات / المرحلة الأولى / الكورس الثاني

شكل المقام	التعبير	شكل الكسور الجزئية
1- liner Factors	$\frac{F(s)}{(s-a)(s+b)}$	$\frac{A}{(s-a)}+\frac{B}{(s+b)}$
2- Repeated linear Factors	$\frac{F(s)}{(s\pm c)^n}$	$\frac{A_1}{(s\pm c)}+\frac{A_2}{(s\pm c)^2}+\cdots+\frac{A_n}{(s\pm c)^n}$
3- quadartic Factors	$\frac{F(s)}{(s^2+as+b)(s^2+cs+d)}$	$\frac{As+B}{s^2+as+b}+\frac{Cs+D}{s^2+cs+d}$
4- Repeated quadartic Factors	$\frac{F(s)}{(s^2+as+b)^n}$	$\frac{A_1s+B_1}{(s^2+as+b)}+\frac{A_2s+B_2}{(s^2+as+b)^2}+\cdots+\frac{A_ns+B_n}{(s^2+as+b)^n}$
5- Mixed Factors	$\frac{F(s)}{(s+a)^2(s^2+bs+c)}$	$\frac{A}{s+a}+\frac{B}{(s+a)^2}+\frac{Cs+D}{s^2+bs+c}$

ex 1: $\mathcal{L}^{-1}\dfrac{4s-3}{s^2-4s-5}$

Sol: $\dfrac{4s-3}{(s-5)(s+1)} = \dfrac{A}{s-5} + \dfrac{B}{s+1}$

$\therefore\ A(s+1) + B(s-5) = 4s-3$

$As + A + Bs - 5B = 4s - 3$

$A + B = 4$

$-A \pm 5B = \mp 3$

$B + 5B = 7 \Rightarrow 6B = 7 \Rightarrow B = \dfrac{7}{6}$

$A = 4 - \dfrac{7}{6} = \dfrac{24-7}{6} = \dfrac{17}{6}$

$\therefore\ \mathcal{L}^{-1}\left(\dfrac{17/6}{s-5} + \dfrac{7/6}{s+1}\right)$

$= \dfrac{17}{6} e^{5t} + \dfrac{7}{6} e^{-t}$

or $\mathcal{L}^{-1}\dfrac{4s-3}{s^2-4s-5} = \mathcal{L}^{-1}\dfrac{4s-3}{s^2-4s+4-4-5} = \mathcal{L}^{-1}\dfrac{4s-3}{(s-2)^2-9}$

$= \mathcal{L}^{-1}\dfrac{4s-8+8-3}{(s-2)^2-9} = \mathcal{L}^{-1}\left(\dfrac{4(s-2)}{(s-2)^2-9} + \dfrac{5}{(s-2)^2-9}\right)$

$= 4e^{2t}\cosh 3t + \dfrac{5}{3} e^{2t} \sinh 3t$

ex_2 / find $\mathcal{L}^{-1} \frac{-5s^2-7s-8}{s^3+3s^2-4s}$

Sol /

$$\frac{-5s^2-7s-8}{s^3-s^2-4s} = \frac{-5s^2-7s-8}{s(s^2+3s-4)} = \frac{-5s^2-7s-8}{s(s+4)(s-1)}$$

$$= \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}$$

$$-5s^2-7s-8 = A(s+4)(s-1) + B\,s(s-1) + C\,s(s+4)$$

let $s=0 \Rightarrow A = \frac{-8}{-4} = 2$

let, $s=-4 \Rightarrow -80+28-8 = 20B \Rightarrow B = -3$

let $s=1 \Rightarrow -5-7-8 = 5C \Rightarrow C = -4$

$$\mathcal{L}^{-1} \frac{2}{s} - \frac{3}{(s+4)} - \frac{4}{(s-1)} = 2 - 3e^{-4t} - 4e^{t}$$

Ex_3 / Find $\mathcal{L}^{-1} \dfrac{4s^2+5}{(s-2)^3}$

Sol/

$$\frac{4s^2+5}{(s-2)^3} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$4s^2+5 = A(s-2)^2 + B(s-2) + C \quad \dots \ (*)$$

let $s = +2 \Rightarrow 4(2)^2+5 = C$

$\Rightarrow C = 21$

to find A, B defferentiate in eq (*) with respect to (s)

$$8s = 2A(s-2) + B \quad \dots \ (**)$$

let $s=2 \Rightarrow B = 16$

To find (A) deferentiate eq (**)

$$8 = 2A \Rightarrow A = 4$$

$$\mathcal{L}^{-1} \frac{4s^2+5}{(s-2)^3} = \mathcal{L}^{-1}\left(\frac{4}{s-2} + \frac{16}{(s-2)^2} + \frac{21}{(s-2)^3} \right)$$

$$= 4e^{2t} + 16te^{2t} + 21e^{2t}\left(\frac{1}{2}t^2\right)$$

$$= 4e^{2t} + 16te^{2t} + \frac{21}{2}t^2e^{2t}$$

$$\mathcal{L}\, t^n = \frac{n!}{s^{n+1}}$$

~~let n = n - 1~~

$$\mathcal{L}\, t^{n-1} = \frac{(n-1)!}{s^n}$$

$$\frac{1}{(n-1)!}\mathcal{L}\, t^{n-1} = \frac{1}{s^n}$$

$$\mathcal{L}^{-1} \frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$$

ex / $\mathcal{L} \dfrac{16}{(s-2)^2}$

let $F(s) = \dfrac{1}{s^2} \Rightarrow \mathcal{L}^{-1} F(s) = F(t)$

$n = 2 \Rightarrow \mathcal{L}^{-1} \dfrac{1}{s^2} = t$

$$\mathcal{L}^{-1} F(s+\alpha) = e^{-\alpha t} F(t)$$

$$\mathcal{L}^{-1} \frac{1}{(s-2)^2} = e^{2t}\, t$$

$$\mathcal{L}^{-1} \frac{16}{(s-2)^2} = 16\, e^{2t}\, t$$

Properties of Laplace Transform

1) $\mathcal{L}\, t\, F(t) = \frac{-d}{ds} f(s)$

ex: $\mathcal{L}\, t^2 \sin 2t$

Sol:

$F(t) = \sin 2t$

$F(s) = \frac{2}{s^2+4}$

$\mathcal{L}\, t\, F(t) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

$= -2(-2s)\frac{1}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$

Let $f(t) = t^2 \sin 2t$

$\mathcal{L}\, t^2 \sin 2t = -\frac{d}{ds}\left[\frac{4s}{(s^2+4)^2}\right]$

$= -4\left[\frac{(s^2+4)^2 - 4(s^2+4)s^2}{(s^2+4)^4}\right] = -4\,\frac{(s^2+4)^2 - 4(s^2)}{(s^2+4)^3}$

$= -4\left[\frac{4-3s^2}{(s^2+4)^3}\right] = \frac{12s^2-16}{(s^2+4)^3}$

ex: find $\mathcal{L}^{-1} \ln \dfrac{s+1}{s-1}$ [illegible]

Sol: $\mathcal{L}\, t\, F(t) = \dfrac{-d}{ds}\left[\ln \dfrac{s+1}{s-1}\right]$

$= -\dfrac{d}{ds}\left[\ln(s+1) - \ln(s-1)\right]$

$= \dfrac{-1}{s+1} + \dfrac{1}{s-1}$

$\mathcal{L}\, t\, F(t) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$

$t \cdot F(t) = \mathcal{L}^{-1} \dfrac{1}{s-1} - \mathcal{L}^{-1} \dfrac{1}{s+1}$

$t\, F(t) = e^{t} - e^{-t}$

$\Rightarrow F(t) = \dfrac{e^{t} - e^{-t}}{t}$

2) laplace Transform of Integral of a function :-

$$\mathcal{L} \int_0^t F(t)\,dt = \frac{1}{s} f(s)$$

ex: Find $\mathcal{L}^{-1} \frac{1}{s(s^2+4)}$

Sol, Let $f(s) = \frac{1}{s^2+4}$

$F(t) = \frac{1}{2} \sin 2t$

$\mathcal{L} \int_0^t (\frac{1}{2} \sin 2t) = \frac{1}{s} (\frac{1}{s^2+4})$

$= \frac{1}{s} (\frac{1}{s^2+4}) = -\frac{1}{4} \cos 2t]_0^t$

$\Rightarrow [\frac{1}{4} \cos 2t]_0^t = \mathcal{L}^{-1} \frac{1}{s(s^2+4)}$

$\mathcal{L}^{-1} \frac{1}{s(s^2+4)} = \frac{1}{4} (1 - \cos 2t)$

3) Integration of Transform :-

$$\mathcal{L}\ \frac{F(t)}{t} = \int_s^{\infty} f(s)\ ds$$

ex: find $\mathcal{L}\ \frac{\sin 2t}{t}$

Sol: Let $f(t) = \sin 2t$

$$f(s) = \frac{2}{s^2+4}$$

$$\mathcal{L}\ \frac{\sin 2t}{t} = \int_s^{\infty} \frac{2}{s^2+4}\ ds$$

$$= \left[\tan^{-1} \frac{s}{2}\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

ex: Find $\mathcal{L}^{-1} \frac{s}{(s^2-1)^2}$

Sol: $\frac{F(t)}{t} = \int_s^{\infty} \frac{s}{(s^2-1)^2} \, ds$

$= \frac{-1}{2} \cdot \frac{1}{s^2-1} \Big|_s^{\infty}$

$= \frac{1}{2} \cdot \frac{1}{(s^2-1)} = \frac{1}{2} \cdot \frac{1}{(s-1)} \cdot \frac{1}{(s+1)}$

$= \frac{1}{2} \left[\frac{1}{s-1} - \frac{1}{s+1} \right] * \frac{1}{2}$

$= \frac{1}{4} \left[\frac{1}{s-1} - \frac{1}{s+1} \right]$

$F(t) = \frac{t}{4} (e^{t} - e^{-t})$

4) laPlace Transform of Derivative of function

$$\mathcal{L} \frac{d}{dt} F(t) = s F(s)$$

ex: Find $\mathcal{L}^{-1} \dfrac{s}{s^2+4}$

Sol:

Let $f(s) = \dfrac{1}{s^2+4}$

$F t = \dfrac{1}{2} \sin 2t$

$$\mathcal{L}^{-1} \frac{s}{s^2+4} = \frac{1}{2} \frac{d}{dt} \sin 2t$$

$$= \cos 2t$$

5) Convolution Theorm :-

If $f(s) = g(s) \cdot h(s)$

$$f(t) = \mathcal{L}^{-1}\, g(s) \cdot h(s)$$

$$= \int_0^t g(\tau)\, h(t-\tau)\, d\tau$$

* تحويلات مهمة

$$\sin(a) \cdot \sin(b) = \frac{1}{2}\left(\cos(\text{فرق}) - \cos(\text{جمع})\right)$$

$$\cos(a) \cdot \cos(b) = \frac{1}{2}\left(\cos(\text{فرق}) + \cos(\text{جمع})\right)$$

$$\sin(a) \cdot \cos(b) = \frac{1}{2}\left(\sin(\text{فرق}) + \sin(\text{جمع})\right)$$

Ex; Find $\mathcal{L}^{-1} \dfrac{s^2}{(s^2+1)^2}$

Sol:

$$\frac{s^2}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{s}{s^2+1} = g(s) \cdot h(s)$$

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{s}{s^2+1} \cdot \frac{s}{s^2+1} = \int_0^t \cos\tau \cdot \cos(t-\tau)\, d\tau$$

$$\cos x \cdot \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y) \right)$$

$$\cos\tau \cos(t-\tau) = \frac{1}{2}\left[\cos(\tau + t - \tau) + \cos(\tau - t + \tau) \right].$$

$$= \frac{1}{2}\left[\cos t + \cos(2\tau - t) \right]$$

$$\Rightarrow \int_0^t \cos\tau \cos(t-\tau)\, d\tau = \frac{1}{2}\int_0^t \cos t + \cos(2\tau - t)$$

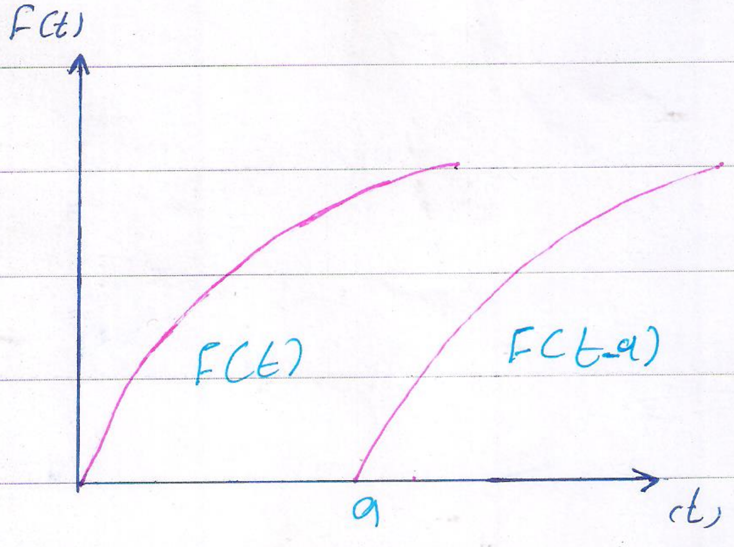
$$= \frac{1}{2} t\cos t + \frac{1}{4}\sin t + \frac{1}{4}\sin t$$

$$\mathcal{L}^{-1} \frac{s^2}{(s^2+1)^2} = \frac{1}{2}\left(t\cos(t) + \sin(t) \right)$$

6) Shifting of the function

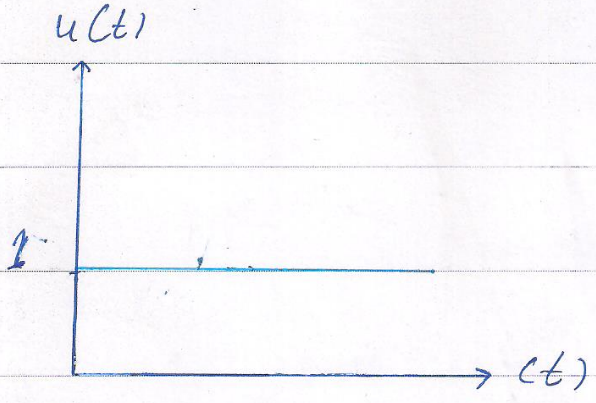
$$\mathcal{L}\, F(t-a)\, u(t-a) = e^{-as} f(s) \quad *$$

$$\mathcal{L}\, F(t)\, u(t-a) = e^{-as} f(t-a) \quad **$$



7) unit step Function:-

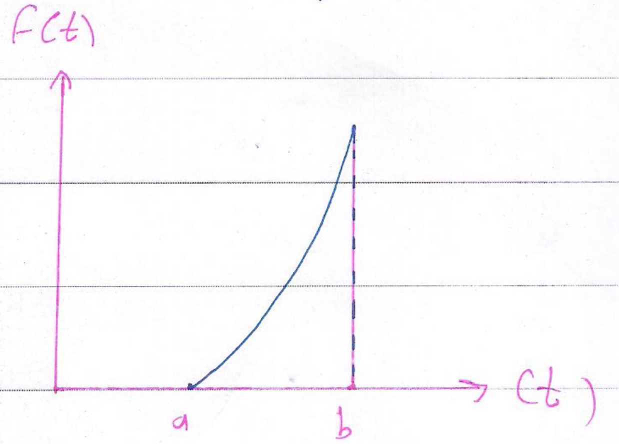
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



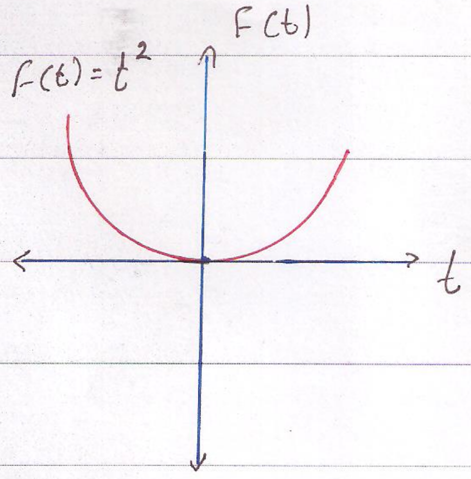
$$\mathcal{L}\, u(t) = \frac{1}{s}$$

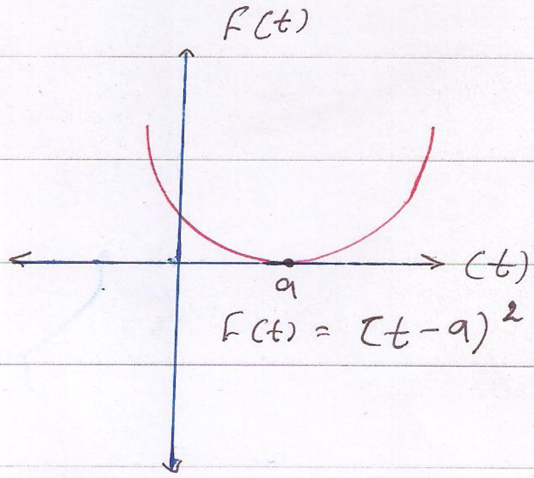
* step Function can be used to find The equation of others. Function.

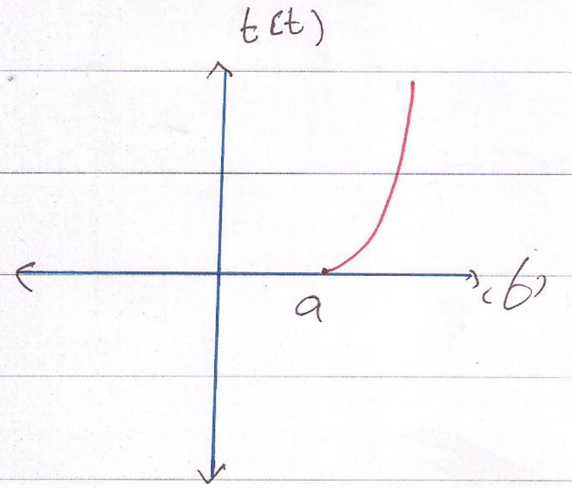
ex: Find the equation of the following curve

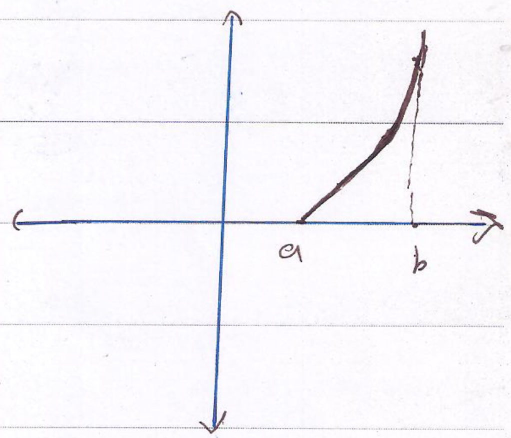


Sol: the above curve can be derived from the following curves :-





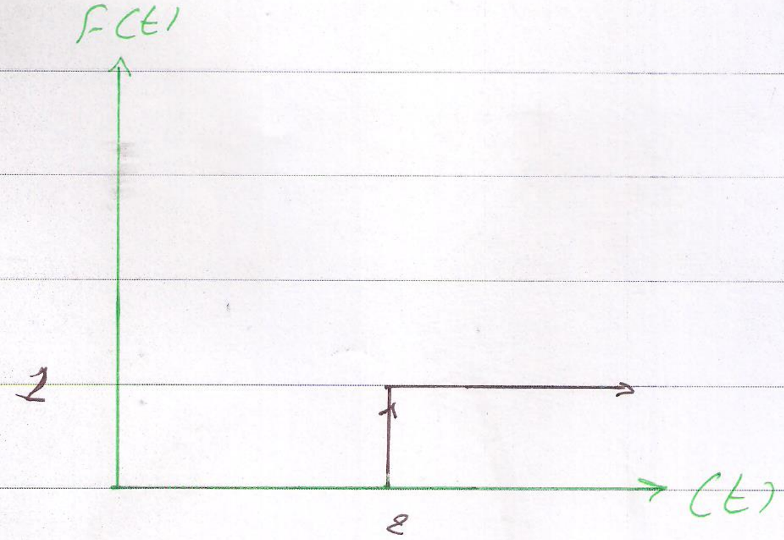




$f(t) = (t-a)^2 u(t-a)$ $\qquad$ $f(t) = (t-a)^2 [u(t-a) - u(t-b)]$

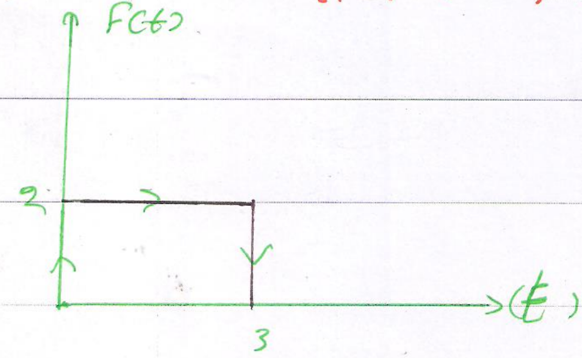
نقطة بداية $\qquad$ نقطة نهاية

ex: Find $\mathcal{L}\, u(t-2)$



Sol: $\mathcal{L}\, u(t-2) = \frac{1}{s} e^{-2s}$

ex: Find the equation and transform of the following graph..



Sol/

$$f(t) \begin{cases} 0 & t \le 0 \\ 2 & 0 \le t \le 3 \\ 0 & t \ge 3 \end{cases}$$

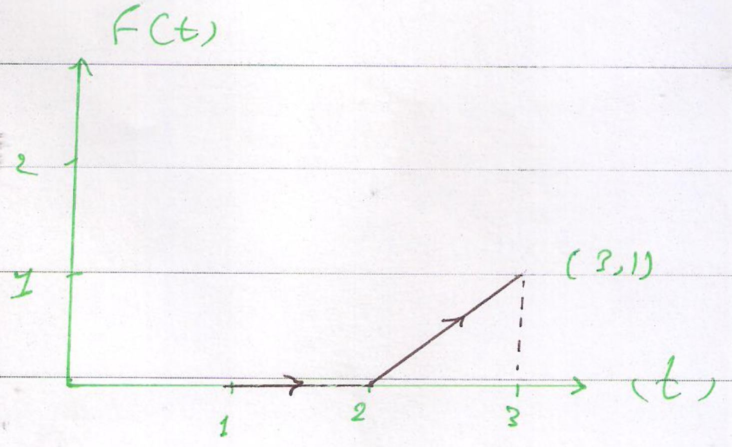
$F(t) = 2[u(t-0) - u(t-3)]$

$F(t) = 2u(t) - 2u(t-3)$

$\mathcal{L}\, f(t) = F(s) = \frac{2}{s} - \frac{2}{s} e^{-3s}$

$= \frac{2}{s}\left[1 - e^{-3s}\right]$

ex: Find equation and Laplace of the curve



$y = mx + c \qquad (0,2)$

$0 = m(2) + c \Rightarrow c = -2m$

$1 = m(3) + c \qquad (3,1)$

$1 = 3m - 2m \Rightarrow m = 1 \Rightarrow c = -2$

$y = x - 2$ (معادلة المستقيم) $\Rightarrow f(t) = t - 2$

$f(t) = t - 2\,[u(t-2) - u(t-3)]$

$f(t) = (t-2)\,u(t-2) - (t-2)\,u(t-3)$

$= (t-2)\,u(t-2) - (t-2+1-1)\,u(t-3)$

$= (t-2)\,u(t-2) - (t-3)\,u(t-3) - u(t-3)$

$\mathcal{L}\,f(t) = \frac{1}{s^2}e^{-2s} - \frac{1}{s^2}e^{-3s} - \frac{1}{s}e^{-3s}$

Ex: Plot the function having the following transfer function.

$$F(s) = \frac{1}{s^2} e^{-2s} - \frac{1}{s^2} e^{-3s} - \frac{1}{s} e^{-3s}$$

* عكس المثال السابق.

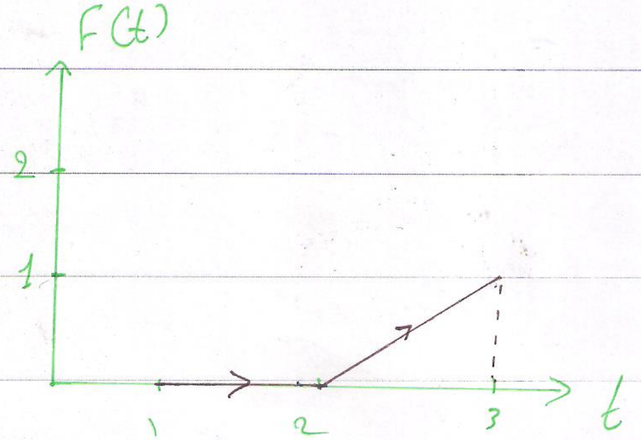
Sol/

$$f(t) = (t-2)u(t-2) - (t-3)u(t-3) - u(t-3)$$

$$f(t)_1 = t-2 \qquad 2 \le t \le 3$$

$$f(t)_2 = t-2-(t-3)-1$$

$$= t-2-t+3-1 = 0 \qquad \langle\langle t > 3 \rangle\rangle$$



8) Transform of Periodic Function:-

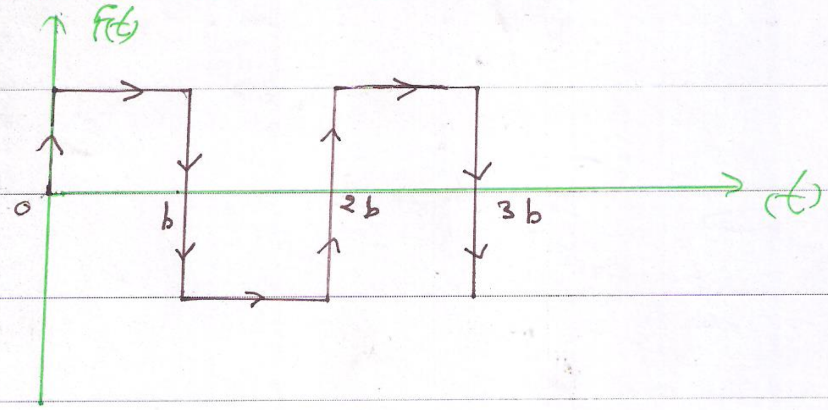
$$\mathcal{L}\, F(t) = f(s) = \frac{1}{1-e^{-as}} \int_0^a F(t)\, e^{-st}\, dt$$

where: $a =$ Period (الفترة)

* الحل تطبيق مباشر * يعرف نوع السؤال من تكرار الدورات

* الفترات مسبق وأن تم دراستها في مادة الكهرباء / الكورس الاول / مرحلة 2.

ex: Find laplace transform of the following Peridice function :-



Sol/:-

$$F(t) = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \\ 1 & 2b < t < 3b \end{cases}$$

$a = 2b$

$$\mathcal{L}\, F(t) = f(s) = \frac{1}{1-e^{-2bs}} \int_0^{2b} F(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^{b} (1)\, e^{-st}\, dt + \int_b^{2b} (-1)\, e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{-1}{s} e^{-st} \right]_0^{b} + \left[\frac{1}{s} e^{-st} \right]_0^{2b}$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{1-e^{-sb}}{s} + \frac{e^{-2bs}-e^{-bs}}{s} \right]$$

$$f(s) = \frac{1}{1-e^{-2bs}} \left[\frac{1-2e^{-sb}+e^{-2bs}}{s} \right]$$

$$= \frac{(1-e^{-bs})^2}{s(1-e^{-2bs})} = \frac{(\cancel{1-e^{-bs}})^2}{s(\cancel{1-e^{-bs}})(1+e^{-bs})}$$

$$= \frac{(1-e^{-bs})}{s(1+e^{-bs})} = \frac{\cancel{e^{\frac{bs}{2}}}(1-e^{-bs})}{s\,\cancel{e^{\frac{bs}{2}}}(1+e^{-bs})}$$

$$= \frac{e^{\frac{bs}{2}} - e^{-\frac{bs}{2}}}{s(e^{\frac{bs}{2}}+e^{-\frac{bs}{2}})} = \frac{1}{s} \tanh\left(\frac{bs}{2}\right)$$

Solution of Differential equation by laplace transform

*طريقة الحل:

1- نحول كل المعادلة بدلالة (S) أي laplace Invers

2- بالنسبة لـ $\frac{dy}{dt}$ ، $\frac{d^2y}{dt^2}$ ، $\frac{d^2y}{dt^2}$ ، $\frac{d^ny}{dt^n}$ نحول كل منها إلى

$$\frac{dy}{dt} \Rightarrow S\,y(s) - y(0)$$

$$\frac{d^2y}{dt^2} \Rightarrow S^2 y(s) - S\,y(0) - y'(0)$$

$$\frac{d^3y}{dt^3} \Rightarrow S^3 y(s) - S^2 y(0) - S\,y'(0) - y''(0)$$

$$\frac{d^ny}{dt^n} = S^n y(s) - S^{n-1} y(0) - S^{n-2} y'(0) \dots$$

$$\dots - S \cdot y^{n-2}(0) - y^{n-1}(0)$$

* $y(0)$، $y'(0)$، $y''(0)$ ... تعطى في السؤال

3- بعدها نحل المعادلة للحصول على $y(t)$

4- احياناً تعطى قيم y ، t .

ex: Solve $\frac{dy}{dt} - 4y = e^{2t}$

where $y = 5$ at $t = 0$

sol/ $s\,y(s) - y(0) - 4\,y(s) = e^{2t}$

$\Rightarrow s\,y(s) - 5 - 4\,y(s) = e^{2t}$

$\Rightarrow y(s)(s-4) = e^{2t} + 5$

$$y(s) = \frac{1}{(s-2)(s-4)} + \frac{5}{s-4}$$

$$\frac{1}{(s-2)(s-4)} = \frac{A}{(s-2)} + \frac{B}{(s-4)} \Rightarrow \begin{array}{l} A = -\frac{1}{2} \\ B = \frac{1}{2} \end{array}$$

$$y(s) = \frac{-1}{2(s-2)} + \frac{1}{2(s-4)} + \frac{5}{(s-4)}$$

$$y(t) = \frac{-1}{2} e^{2t} + \frac{1}{2} e^{4t} + 5 e^{4t}$$

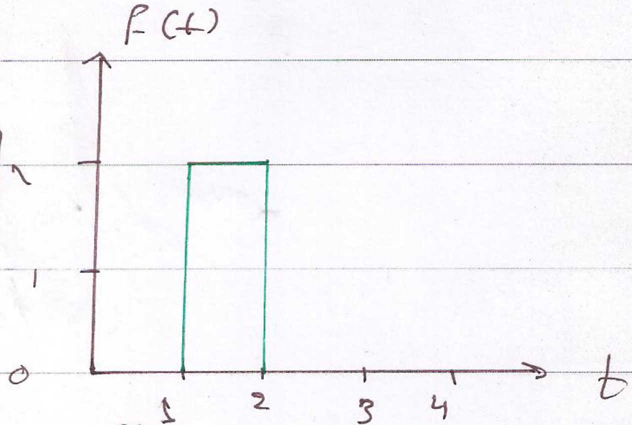
$$\Rightarrow y(t) = 11/2\, e^{4t} - \frac{1}{2} e^{2t}$$

ex: Solve: $\dfrac{dy}{dt} + 3y + 2\int_0^t y\,dt = f(t)$

where $f(t) = \begin{cases} 0 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 0 & 2 < t \end{cases}$

and $y(0) = 1$

Sol /



$F(t) = 2[u(t-1) - u(t-2)]$

$= 2u(t-1) - 2u(t-2)$

$F(s) = \dfrac{2}{s} e^{-s} - \dfrac{2}{s} e^{-2s}$

$sY(s) - y(0) + 3Y(s) + \dfrac{2}{s} Y(s) = \dfrac{2}{s} e^{-s} - \dfrac{2}{s} e^{-2s}$

$\Rightarrow Y(s)\left(s + 3 + \dfrac{2}{s}\right) = 1 + \dfrac{2}{s} e^{-s} - \dfrac{2}{s} e^{-2s} \quad (\times s)$

$\Rightarrow Y(0)(s^2 + 3s + 2) = s + 2e^{-s} - 2e^{-2s}.$

$\Rightarrow Y(s) = \dfrac{1}{(s+2)(s+1)} \left[s + 2e^{-s} - 2e^{-2s} \right]$

$$y(s) = \frac{s}{(s+2)(s+1)} + \frac{2}{(s+2)(s+1)} e^{-s} - \frac{2e^{-2s}}{(s+2)(s+1)}$$

$$\frac{s}{(s+2)(s+1)} = \frac{A}{(s+2)} + \frac{B}{(s+1)} \Rightarrow \begin{matrix} A = 2 \\ B = -1 \end{matrix}$$

$$\frac{2}{(s+2)(s+1)} = \frac{A_2}{(s+2)} + \frac{B_2}{(s+1)} \Rightarrow \begin{matrix} A_2 = -1 \\ B_2 = +1 \end{matrix}$$

$$y(s) = \frac{2}{s+2} - \frac{1}{s+1} + \left(\frac{2}{(s+1)} - \frac{2}{(s+2)}\right) e^{-s}$$

$$- \left(\frac{2}{(s+1)} - \frac{2}{(s+2)}\right) e^{-2s}$$

$$\Rightarrow y(t) = 2e^{-2t} - e^{-t} + \left[2e^{-(t-1)} - 2e^{-2(t-1)}\right]$$

$$u(t-1) - \left[2e^{-(t-2)} - 2e^{-2(t-2)}\right]$$

$$u(t-1)$$

Ex: Solve :- $y'' + 2y' + y = t e^{-t}$

$y(0) = 1$ and $y'(0) = -2$

Sol,

$$s^2 y(s) - s y(0) - y'(0) + 2s y(s) - 2y(0)$$
$$+ y(s) = -\frac{d}{ds}\left[\frac{1}{s+1}\right]$$

$$\Rightarrow s^2 y(s) - s - 2 + 2s y(s) - 2 + y(s) = \frac{1}{(s+1)^2}$$

$$\Rightarrow y(s)(s^2 + 2s + 1) - s + 2 - 2 = \frac{1}{(s+1)^2}$$

$$\Rightarrow y(s) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$\Rightarrow f(t) = \frac{t^3}{6} e^{-t} + e^{-t} - t e^{-t}$$

Solution of Simultaneous

Differential equation by laplace Transfer:-

Ex: Solve: $y'' + z + y = 0$ --- (1)

$y' + z' = 0$ --- (2)

where $y(0) = 0$, $y'(0) = 0$, $z(0) = 1$

Sol/ $s^2(y(s)) - s\,y(0) - y'(0) + z(s) + y(s) = 0$ --- (1)

$s\,y(s) - y(0) + s\,z(s) - z(0) = 0$ --- (2)

$s^2 y(s) + z(s) + y(s) = 0$ --- (3)

$s\,y(s) + s\,z(s) - 1 = 0$ --- (4)

(3) $\Rightarrow y(s) = \dfrac{-1}{s^2+1}\,z(s)$, (4) $\Rightarrow y(s) = \dfrac{1}{s} - z(s)$

sub (4) in (3) $\Rightarrow y(s) = \dfrac{-1}{s^2+1}\left[\dfrac{1}{s} - y(s)\right]$

$\Rightarrow y(s)\left(1 - \dfrac{1}{s^2+1}\right) = \dfrac{-1}{s(s^2+1)}$

$s^2 y(s) = \dfrac{-1}{s} \Rightarrow y(s) = -\dfrac{1}{s^2} \Rightarrow$ $y(t) = \dfrac{-1}{2}t^2$

$z(s) = \dfrac{1}{s} + \dfrac{1}{s^2} \Rightarrow$ $z(t) = 1 + \dfrac{1}{2}t^2$

The end